MATHEMATICS

A Textbook for Class XII

Part I

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The top of the front cover shows simple computations in ancient Egyptian, Babylonian and Greek numerals These and the drawing of a computer circuit at the centre suggest the evolution of mathematical thinking over centuries.

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Foreword

The National Policy on Education (NPE) 1986 has emphasised the need for qualitative improvement of school education, particularly in the area of science and mathematics education. The Government of India has already initiated a number of steps in this direction. The National Council of Educational Research and Training (NCERT) has been assigned the responsibility of developing a new curriculum and related curricular materials in line with the new education policy to serve as models for the States and the Union Territories to adopt/adapt.

The NCERT has been working in the area of curriculum development/renewal for many years. Yet developing a curriculum in tune with the intentions and aspirations of NPE poses a real challenge. The various curricular issues arising out of the NPE were discussed in a number of seminars/workshops resulting in the Council's document *National Curriculum for Elementary and Secondary Education—A Framework* (Revised version 1987) and several other documents.

Science and mathematics are the vital areas of school curriculum. So, the NCERT thought it appropriate to make the best use of the available expertise in the country in science and mathematics education in the process of curriculum renewal and preparation of new generation of instructional materials. Accordingly, a General Advisory Board for Science and Mathematics was constituted under the Chairmanship of Professor C.N.R. Rao, an eminent scientist and Chairman of the Prime Minister's Science Advisory Council. On the advice of this General Advisory Board, six writing teams were set up for developing new instructional packages in science and mathematics in line with the NPE. The writing team for mathematics was constituted under the chairmanship of Prof. U.N. Singh. The team consists of distinguished mathematicians from various universities, besides NCERT experts. The writing team met several times and after much deliberations evolved a new mathematics curriculum which, I am sure reflects the intentions and aspirations of the NPE. In the previous year, the writing team developed the textbook for Class XI based on the new syllabus. The present textbook for Class XII, is in continuation of the textbook for Class XI.

The authors spared no efforts to produce materials of high quality. First, the draft materials prepared by different authors were continuously refined through mutual discussions within the group. Then the materials were exposed to a group of classroom teachers in a review workshop. The suggestions and comments made in the review workshop were incorporated by the authors as far as possible and finally the whole manuscript was edited by Prof S. Izhar Husain in consultation with the other members of his team

I am indeed very thankful to Prof S Izhar Husain, Prof. M.S Rangachari, Prof. V. Kannan, Prof D.D Joshi, Prof. U B. Tewari, Prof. V.G. Tikekar, Prof K.V Rao who authored different parts of the book I am particularly grateful to Prof S. Izhar Husain who edited the whole manuscript. I thank Prof. U.N Singh, chairman of the writing team, for his guidance in the development of this book. I express my deep appreciation to my colleagues in the Department of Education in Science and Mathematics, Prof K.V Rao, Prof S C Das, Dr B. Deokinandan, Dr Hukum Singh and Dr (Mrs) Mukti Acharya who took a lot of pains in shaping the manuscript in the pressworthy form and seeing it through the press I am very much indebted to the teachers who participated in the review workshop and provided valuable suggestions and comments for the improvement of the draft materials I thank Prof B Ganguly, Head, Department of Education in Science and Mathematics, who took a lot of interest in this project and greatly helped in bringing out this book. I also express my thanks to Shri C N. Rao, Head, Publication Department, and his Publication Team for making all efforts in bringing out this book expeditiously and in an excellent form.

Curriculum development is a dynamic and continuous process. No one can claim to have developed a perfect curriculum or perfect curricular materials. Though Prof. U.N Singh and his able team of mathematicians did a very good job of evolving a new mathematics curriculum and new curricular materials in line with the NPE, there can always be some scope for further improvement So, I request all those who will be using this book to offer their valuable suggestions for further improvement.

P.L MALHOTRA
Director
National Council of Educational
Research and Training

Preface

As I had said in the Preface to the textbook for Class XI, the new series of textbooks in mathematics being prepared by the NCERT has been written in the light of paragraphs 8 16 and 8 17 of the National Policy on Education 1986 These paragraphs are quoted in full in the Preface to the textbook for Class XI. The textbook for Class XII has been written in continuation of the textbook for Class XI. Most of the topics introduced in the book for Class XI have been further studied in the present textbook as a logical development of the subject matter studied in Class XI I reproduce below a part of my Preface to the textbook for Class XI with slight modifications, because the points emphasized there are equally applicable to the present textbook.

The implementation of the NPE made it necessary to review the courses of studies in science and mathematics and to bring out new textbooks. There is another strong reason for reviewing the course contents in science and mathematics and for rewriting textbooks in these disciplines Different branches of science, pure and applied, including mathematics are developing with astonishing rapidity Exciting discoveries of far-reaching importance are being made in quick succession. Deep and new ideas of a rapidly growing science very often shed new penetrating light even on the most elementary topics. It is, therefore, highly desirable that courses of studies of school education are reviewed periodically and new textbooks are written.

The National Council of Educational Research and Training initiated prompt action in respect of redesigning the curricula in science subjects and mathematics relating to school education. It appointed a General Advisory Board for Science and Mathematics under the chairmanship of Professor C.N.R. Rao, who is also the Chairman of the Science Advisory Council of the Prime Minister. On the advice of General Advisory Board, the NCERT constituted six writing teams for developing instructional packages in science subjects and mathematics from upper primary to senior secondary level. Besides the experts from the NCERT, distinguished mathematicians from different parts of the country are members of the writing team in mathematics.

The NCERT had done a good deal of preparatory work in connection with curriculum renewal before the appointment of the writing team. We have been greatly benefited in our work by the NCERT's documents :

- National Curriculum for Elementary and Secondary Education — A Framework
- Mathematics education for the first 10 years of schooling — Guidelines for

developing curriculum for upper primary and high school stages.
- Draft syllabi in mathematics for upper primary, secondary and senio secondary levels.

The present textbook has been written on the basis of a curriculum which emerged after a thorough review of the curriculum prepared by the NCERT. A new textbook should be written only when it has to say new things or give a new message. I believe that the present book has some new ideas Some special features of this book are as follows :

1 This book has also tried to lay strong foundations for the study of mathematics as a discipline For this purpose some of the useful basic concepts of higher mathematics have been introduced in a simple way.
2 Problem-solving forms an important part of mathematical training This book has a good number of challenging problems for the talented, besides problems for drill, understanding application, etc
3 Computers are going to play a vital role in our lives. Some computer related ideas have already been introduced in the earlier books In this book, elementary ideas of computer programming are introduced in a simple way
4. The chapters on mathematical logic and numerical methods have been written in a simple style to introduce some 'computer-oriented' mathematics.
5. New concepts have been introduced by means of simple examples.

Our group is also working on the development of additional instructional materials to supplement the textbook The additional materials are Supplementary Problem Book, Enrichment Mathematics, Teachers' Guide, and so on. I hope these additional materials will soon be made available to the students and teachers

I am thankful to Dr P.L. Malhotra, Director, NCERT who initiated this project and invited us to join this national endeavour for the improvement of mathematics education.

I am grateful to Prof C N R. Rao for his constant guidance which helped us in planning and developing this textbook I express my sincere thanks to Prof. A.K. Jalaluddin, Joint Director, NCERT, and Prof B. Ganguly, Head, Department of Education in Science and Mathematics, for their kind cooperation extended to me and to the writing team.

Prof S. Izhar Husain, Prof M S Rangachari, Prof. D.D. Joshi, Prof. V Kannan, Prof. V G Tikekar, Prof. U.B. Tewari, Prof K.V Rao, developed the materials of this book The first drafts developed by the authors were first discussed among the authors and revised The revised drafts were then discussed with the school teachers in a workshop organised by the NCERT. The materials were modified in the light of the suggestions made by the teachers. Finally, the chapters were edited by Prof Izhar Husain, convenor of the group in consultation with other members of the team. Thus, the present textbook is a product of the combined thinking and efforts of several persons involved in the process I express my deep sense of gratitude to these colleagues of mine for their valuable contribution to the development of this book. I am also thankful to the school teachers who attended the workshop at NCERT, New Delhi and made valuable suggestions for improving the materials. If the present series of textbooks contributes to the improvement of the quality of mathematical education in schools, our efforts will be amply remunerated.

Prof. K.V. Rao, Prof. S C. Das, Dr B Deokinandan, Dr Hukum Singh, Dr (Mrs) Mukti Acharya of NCERT had to put in hard work in organising several workshops, getting the manuscript in pressworthy form and finally seeing it through the press. I am indeed very thankful to them.

In spite of the great care taken by the authors and the NCERT team, some errors may have escaped our notice We shall appreciate it very much if such errors are brought to our notice. Suggestions for improving the quality of the book will be gratefully received.

U.N. SINGH
Chairman of the Writing Team

गांधी जी का जन्तर

तुम्हें एक जन्तर देता हूं। जब भी तुम्हें सन्देह हो या तुम्हारा अहम् तुम पर हावी होने लगे, तो यह कसौटी आजमाओ :

जो सबसे गरीब और कमजोर आदमी तुमने देखा हो, उसकी शकल याद करो और अपने दिल से पूछो कि जो कदम उठाने का तुम विचार कर रहे हो, वह उस आदमी के लिए कितना उपयोगी होगा। क्या उससे उसे कुछ लाभ पहुंचेगा? क्या उससे वह अपने ही जीवन और भाग्य पर कुछ काबू रख सकेगा? यानि क्या उससे उन करोड़ों लोगों को स्वराज्य मिल सकेगा जिनके पेट भूखे हैं और आत्मा अतृप्त है?

तब तुम देखोगे कि तुम्हारा सन्देह मिट रहा है और अहम् समाप्त होता जा रहा है।

मो क गांधी

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CHAPTER 1

MATRICES AND DETERMINANTS

IN THIS CHAPTER we shall learn about matrices and determinants. These have become an important tool in the study of science and engineering. We shall confine ourselves to the study of basic laws of matrix algebra, elementary properties of determinants and application of these concepts in solving a system of linear equations.

1.1 Matrices

Suppose we wish to express that Ram has 10 books. We may express it as [10] with the understanding that the number inside [ ] is the number of books Ram has. Next suppose we want to express that Ram has 10 books and 5 pencils We may express it as [10 5] with the understanding that first number inside [ ] is the number of books while the second number is the number of pencils possessed by Ram. Let us now suppose that we have three children- Ram, Rahim and Gur Bux and Ram has 10 books and 5 pencils, Rahim has 3 books and 4 pencils and Gur Bux has 5 books and 6 pencils. How can we express this information ? This may be done in the tabular form as follows.

	Books	Pencils
Ram	10	5
Rahim	3	4
Gur Bux	5	6

We can also briefly write this as follows :

$$\begin{array}{cc}
\begin{bmatrix} 10 & 5 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} & \begin{array}{l} \leftarrow \text{First row} \\ \leftarrow \text{Second row} \\ \leftarrow \text{Third row} \end{array} \\
\begin{array}{cc} \uparrow & \uparrow \\ \text{First} & \text{Second} \\ \text{Column} & \text{Column} \end{array} &
\end{array}$$

Implied in the above display is the following information :

(i) The entries in the first second and third rows represent the objects (books and pencils) possessed by Ram, Rahim and Gur Bux respectively.

(ii) The entries in the first and second column represent the number of books and the number of pencils respectively.

Thus the entry in the second row and second column represents the number of pencils possessed by Rahim. Similarly, each entry in the above display can be interpreted.

An arrangement or display of the above kind is called a *matrix*. Formally we have the following definition of a matrix .

Definition

A matrix is a rectangular array of numbers.

We denote the matrices by cpital letters.

Following are some examples of matrices :

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2+i & 4 & 7 \\ 1 & -3+i & 6-i \end{bmatrix}$$

$$C + \begin{bmatrix} 1+x & 1-x^2 & -6 \\ -3 & 2 & 0 \\ 7+\sin x & 2-\cos x & 5 \end{bmatrix}$$

In the above examples, the horizontal lines of numbers are called *rows* of the matrix and the vertical lines of numbers are called *columns* of the matrix. Thus A has 2 rows and 2 columns, B has 2 rows and 3 columns, while C has 3 rows and 3 columns A matrix having m rows and n columns is called a matrix of order $m \times n$ or simply a $m \times n$ *matrix* (read as a m by *n matrix*). In general, a $m \times n$ matrix is written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots a_{in} \\ \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}$$

Thus the i th row consists of the entries

$$a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in}$$

while the jth column consists of the entries

$$a_{1j}, a_{2j}, a_{3j}, \ldots, a_{mj} .$$

The numbers a_{ij} which appear in the matrix are called its *elements* or *entries*. The element a_{ij} appears in the i the row and the j th column of the matrix We also call it as the (i, j)th element of A The notation $A = [a_{ij}]$ is also often used The notations $A = (a_{ij})$ is also in use We shall follow the earlier notation, namely $A = [a_{ij}]$.

Example 1.1

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \\ 9 & 11 & 13 \end{bmatrix}$$

Then A is 3×3 matrix. Here, the element '6' occurs in the second row and third column. The element '9' occurs in third row and first column. Thus in the general notation we may write $a_{23} = 6$ and $a_{31} = 9$.

Example 1.2

Consider the following information regarding the number of men and women workers in three factories I, II and III.

	Men workers	Women workers
I	30	5
II	25	11
III	27	6

Represent the above information in the form of a 3×2 matrix. What does the entry in the third row and second column represent ?

Solution

The information is represented in the form of a 3×2 matrix as follows

$$\begin{bmatrix} 30 & 5 \\ 25 & 11 \\ 27 & 6 \end{bmatrix}$$

The entry in the third row and second column represents the number of women workers in factory III.

Example 1.3

Consider the linear equations

$$\begin{aligned} 2x + 3y - 5z &= 4 \\ x + y + 2z &= 5 \end{aligned}$$

Then the coefficients of x, y, z in the above equations can be expressed in the form of the following matrix :

$$\begin{bmatrix} 2 & 3 & -5 \\ 1 & 1 & 2 \end{bmatrix}$$

1.2. Types of Matrices

Definition

A $1 \times n$ matrix is called a *row matrix* or a *row vector.*
For example, $[\,1 \;\; 0 \;\; 5\,]$ is a row matrix of order 1×3.

Definition

A $m \times 1$ matrix is called a *column matrix* or a *column vector.*
The matrices

$$\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 0 \\ 2 \end{bmatrix}$$

are examples of column matrices.

Definition

A matrix having the same number of rows and columns is called a *square matrix.*

Thus, a $m \times n$ matrix A is a square matrix if $m = n$ In this case, we also say that A is of order n.

Definition

The *zero matrix* is the matrix, all of whose elements are zeros.

Thus, $[0 \quad 0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

are all zero matrices, but of different orders. The zero matrix is denoted by the symbol O. Its order will be clear from the context. Zero matrix is also called the *null* matrix.

Definition

The *unit matrix* or the *identity matrix* is the square matrix with 1's on the main diagonal and zeros elsewhere. For example, the matrices

$$[\,1\,], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are identity matrices of order 1, 2 and 3 respectively The square matrix $[a_{ij}]$ is an identity matrix if

$$a_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

The identity matrix is denoted by I and its order will be clear from the context.

Definition

A square matrix having non-zero entries only on the main diagonal is called a *diagonal matrix*

Thus the square matrix $[a_{ij}]$ is a diagonal matrix if

$$a_{ij} = 0 \qquad \text{for } i \neq j$$

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ are examples of diagonal matrices.

Definition

A diagonal matrix with equal non-zero entries on the main diagonal is called a *scalar matrix*.

Thus, the square matrix $[a_{ij}]$ is a scalar matrix if $a_{ij} = \begin{cases} \alpha & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases}$ where α is a number.

For example,

$$[2], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1+i \end{bmatrix}$$

are scalar matrices of order 1, 2 and 3 respectively.

1.3. Operations on Matrices

Definition

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be *equal* if they are of the same order (that is, have the same number of rows and the same number of columns) and each element of A is equal to the corresponding element of B; that is, $a_{ij} = b_{ij}$ for all i, j.

Example 1.4

$[a \quad b] = [1 \quad 2]$ if and only if $a = 1$ and $b = 2$.

Example 1.5

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ if and only if}$$

$$a = 0,\ b = 1,\ c = 1 \text{ and } d = 0.$$

Addition of Matrices

Suppose Mahesh has two factories at places A and B. Each factory produces clothing for boys and girls in three price styles, labelled 1, 2 and 3 The quantities (in standard units) produced by each factory are given in the matrices shown below.

$$\begin{array}{l} \\ \\ \text{Boys} \\ \text{Girls} \end{array} \begin{array}{c} \text{Factory at } A \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 75 & 70 & 72 \\ 60 & 65 & 40 \end{bmatrix} \end{array} \qquad \begin{array}{c} \text{Factory at } B \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 25 & 35 & 40 \\ 30 & 22 & 36 \end{bmatrix} \end{array}$$

Suppose we want to find the total production of clothing in each style. Then the total production in style 1 for boys will be $(75 + 25)$ and for girls will be $(60 + 30)$. Similarly, in style 2 for boys, we will have $(70 + 35)$ and for girls $(65 + 22)$, and in style 3 for boys, we will have $(72 + 40)$ and for girls $(40 + 36)$. This can be represented in matrix form as

$$\begin{bmatrix} 75+25 & 70+35 & 72+40 \\ 60+30 & 65+22 & 40+36 \end{bmatrix}$$

This new matrix is the *sum* of the above two matrices. We observe that the sum of two matrices is a matrix obtained by adding the corresponding elements of the given matrices. Further-more, for the sum to be defined the matrices have to be of the same order. Formally, we have the following definition :

Definition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order, their sum $A + B$ is the matrix whose (i, j)th element is $a_{ij} + b_{ij}$.

Thus, $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

Example 1.6

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 & -6 \\ 2 & 0 & 7 \end{bmatrix}$

$$\text{Then } A+B = \begin{bmatrix} 1+3 & 0+5 & -1-6 \\ 2+2 & 3+0 & 4+7 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & -7 \\ 4 & 3 & 11 \end{bmatrix}$$

Note

We emphasize that if A and B are *not* of the same order, then $A+B$ is *not* defined. For example, if

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix},$$

then $A+B$ is not defined.

Theorem 1 1

Matrix addition is both commutative and associative. We will prove that if A, B, C are three matrices of the same order, then

(i) $A+B=B+A$ (commutativity)

(ii) $(A+B)+C=A+(B+C)$ (Associativity)

Proof : Let $A=[a_{ij}]$, $B=[B_{ij}]$ and $C=[C_{ij}]$. Then

(i) $A+B=([a_{ij}]+[b_{ij}])=[a_{ij}+b_{ij}]$

$=[b_{ij}+a_{ij}]$ (addition of numbers is commutative)

$=([b_{ij}]+[a_{ij}])$

$=B+A$

(ii) $(A+B)+C=([a_{ij}]+[b_{ij}])+[c_{ij}]$

$=[a_{ij}+b_{ij}]+[c_{ij}]$ (Why?)

$=[(a_{ij}+b_{ij})+c_{ij}]$ (Why?)

$=[a_{ij}+(b_{ij}+c_{ij})]$ (addition of numbers is associative)

$=[a_{ij}]+[b_{ij}+c_{ij}]$

$=[a_{ij}]+([b_{ij}]+[c_{ij}])$

$=A+(B+C)$

Scalar Multiplication

Definition

If $A=[a_{ij}]$ and k is a number, then $k\,A$ is the matrix whose (i, j)th element is $k\,a_{ij}$. Thus, $k\,A=[k\,a_{ij}]$

Example 1 7

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 2 \\ 2 & 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Theorem 1.2

If A, B are two matrices of the same order and k and l are numbers, then

(i) $k(A+B) = kA+kB$

(ii) $(k+l)A = kA+lA$

Proof : Let $A=[a_{ij}]$ and $B=[b_{ij}]$

Then $A+B=[a_{ij}+b_{ij}]$

$$\begin{aligned} \text{Hence, } k(A+B) &= [k(a_{ij}+b_{ij})] \\ &= [k\,a_{ij}+k\,b_{ij}] && \text{(distributive property)} \\ &= [k\,a_{ij}]+[k\,b_{ij}] \\ &= k\,A+k\,B \end{aligned}$$

$$\begin{aligned} \text{Also, } (k+l)\,A &= [(k+l)\,a_{ij}] \\ &= [k\,a_{ij}+l\,a_{ij}] && \text{(distributive property)} \\ &= k\,A+l\,A \end{aligned}$$

Remark

Let $A=[a_{ij}]$

Then $(-1)\,A=[(-1)\,a_{ij}]=[-a_{ij}]$.

We write $(-1)\,A$ as $-A$.

It is obvious that $A+(-A)=O$.

Also, [illegible] $=A$.

Thus, O is an additive identity and $-A$ is an additive inverse of A in the set of all $m\times n$ matrices.

Matrix Multiplication

Suppose Ram and Shyam are two friends. Ram wants to buy 3 pencils and 3 notebooks, while Shyam needs 6 pencils and 5 notebooks. They go to a shop and are quoted the following rates :

Pencil—60 paise each

Notebook—120 paise each

How much money does each need to spend ? Clearly, Ram needs $(3\times 60+3\times 120)$ paise, that is, 540 paise, while Shyam needs $(6\times 60+5\times 120)$ paise, that is, 960 paise.

In terms of matrix representation, we can write the above information as follows :

$$\begin{array}{cccc} \text{Requirements} & \text{Prices (in paise)} & \text{Money needed (in paise)} & \\ \begin{bmatrix} 3 & 3 \\ 6 & 5 \end{bmatrix} & \begin{bmatrix} 60 \\ 120 \end{bmatrix} & \begin{bmatrix} 3\times 60+3\times 120 \\ 6\times 60+5\times 120 \end{bmatrix} & = \begin{bmatrix} 540 \\ 960 \end{bmatrix} \end{array}$$

Suppose there is another shop in the locality quoting the following prices :

Pencil—70 paise each
Notebook—130 paise each

Now, the mony needed by Ram and Shyam to make purchases from this shop will be $(3\times70+3\times130)$ paise, that is, 600 paise and $(6\times70+5\times130)$ paise, that is, 1070 paise respectively. We can write this information as follows :

Requirements Prices (in paise) Money needed (in paise) at the two shops

$$\begin{bmatrix} 3 & 3 \\ 6 & 5 \end{bmatrix} \qquad \begin{bmatrix} 70 \\ 130 \end{bmatrix} \qquad \begin{bmatrix} 3\times70+3\times130 \\ 6\times70+5\times130 \end{bmatrix} = \begin{bmatrix} 600 \\ 1070 \end{bmatrix}$$

The above two informations combined, can be expressed in terms of matrices as follows :

Requirements Prices (in paise) Money (in paise) needed at the two shops

$$\begin{bmatrix} 3 & 3 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 60 & 70 \\ 120 & 130 \end{bmatrix} \qquad \begin{bmatrix} 3\times60+3\times120 & 3\times70+3\times130 \\ 6\times60+5\times120 & 6\times70+5\times130 \end{bmatrix} = \begin{bmatrix} 540 & 600 \\ 960 & 1070 \end{bmatrix}$$

The above example illustrates multiplication of matrices. We observe that for multiplication of two matrices A and B, the number of cloumns in A should be equal to the number of rows in B. Furthermore, for getting the elements of the product matrix, we take rows of A and columns of B, multiply them elementwise and take the sum. Below we explain these in detail.

Let A be a $m\times n$ matrix and B a $n\times p$ matrix, that is, the number of columns of A is equal to the number of rows of B The product C of the matrices A and B is a matrix defined as follows · To get (i, j)th element C_{ij} of C, we take the i th row of A and the j the column of B, multiply them elementwise and take the sum of all these products.

Thus if $A = [a_{ij}]$, $1\leq i\leq m$, $1\leq j\leq n$, and

$B = [b_{kl}]$, $1\leq k\leq n$, $1\leq l\leq p$, then the i th row of A is $= [a_{i1}\ a_{i2}\ .....\ a_{in}]$ and the j th column of B is

$$\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

and
$$C_{ij} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \dots + a_{in} b_{nj}$$
$$= \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

Note

1. If the number of columns of A is different from the number of rows of B, the product AB is *not* defined.

2. If A is a $m\times n$ matrix and B is a $n\times p$ matrix, then AB is a $m\times p$ matrix.

Example 1.8

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 4 & 2 \\ 1 & 5 & 3 \end{bmatrix}$

Number of columns of $A = 3$.
Number of rows of $B = 3$.
Hence, AB is defined and is a 2×3 matrix. Also,

$$AB = \begin{bmatrix} 1\times2 + 2\times5 + 3\times1 & 1\times3 + 2\times4 + 3\times5 & 1\times1 + 2\times2 + 3\times3 \\ -2\times2 + 1\times5 + 4\times1 & -2\times3 + 1\times4 + 4\times5 & -2\times1 + 1\times2 + 4\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 26 & 14 \\ 5 & 18 & 12 \end{bmatrix}$$

Remark

If AB is defined, then BA need not be defined This is exhibited by the above example. BA is not defined in the above case because B has 3 columns while A has only 2 (and not 3) rows.

Now, we shall see by an example that even if AB and BA are both defined, it is not necessary that $AB = BA$

Example 1.9

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Now A is a 3×2 matrix and B is a 2×3 matrix. Hence, AB and BA are both defined and are matrices of order 3×3 and 2×2 respectively. $AB \neq BA$ as they are matrices of different orders

One may think that AB and BA will be the same if they are of the same order. However, we give an example below to show that AB and BA may be different even if they are of the same order.

Example 1.10

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$

Then $AB = \begin{bmatrix} 0 & 0 \\ -2 & 2 \end{bmatrix}$ while

$BA = \begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix}$

Hence, $AB \neq BA$

Thus matrix multiplication is not commutative.

For numbers a, b we know that if $ab = 0$, then either $a = o$ or $b = 0$. This need not be true for matrices.

Example 1 11

Let $A = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ and $B = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$

where a, b, c, d are all different from zero. Then $A \neq O$, $B \neq O$,

but $AB = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Hence, if the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

Theorem 1.3

Matrix multiplication is associative and it distributes over addition. In other words,

(1) $A \ (BC) = (AB)\ C$

(2) $A \ (B + C) = AB + \ AC$

(3) $(A + B))\ C = AC + BC$

whenever both sides of the equality are defined.

Proof : (1) Let $A = [a_{ij}]_{(m \times n)}$,
$B = [b_{kl}]_{(n \times p)}$ and $C = [C_{rs}]_{(p \times q)}$

Now, (i, k)th element of AB

$$= \sum_{j=1}^{n} a_{ij}\, b_{jk}$$

Hence, (i, s)th element of $(AB)\ C$

$$= \sum_{k=1}^{p} \Big(\sum_{j=1}^{n} a_{ij}\, b_{jk}\Big)\, C_{ks} \qquad \ldots (1.1)$$

Also (j, s)th element of (BC)

$$= \sum_{k=1}^{p} b_{jk}\, c_{ks}$$

Hence, (i, s)th element of $A\ (BC)$

$$= \sum_{j=1}^{n} a_{ij}\, \Big(\sum_{k=1}^{p} b_{jk}\, c_{ks}\Big) \qquad (1.2)$$

From (1 1) and (1.2), we conclude that

$A\ (BC) = (AB)\ C$.

(2) Let $A = [a_{ij}]_{(m \times n)}$
$B = [b_{kl}]_{(n \times p)}$
$C = [c_{kl}]_{(n \times p)}$

Then $B + C = [b_{kl} + c_{kl}]$

Hence, (i, k)th element of $A\,(B + C)$

$$= \sum_{j=1}^{n} a_{ij}\,(b_{jk} + c_{jk})$$

$$= \sum_{j=1}^{n} a_{ij}\, b_{jk} + \sum_{j=1}^{n} a_{ij}\, c_{jk}$$

= (i, k)th element of AB + (i, k)th element of AC

Therefore, $A\,(B + C) = AB + AC$.

The proof of (3) is exactly similar to that of (2).

Transpose of a Matrix

Let A be a $m \times n$ matrix. Then the matrix A' obtained by interchanging the rows and columns of A is called the transpose of A.

Example 1.12

Let A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then A' = $\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

Note

If A is a $m \times n$ *matrix, then* A' *is a* $n \times m$ *matrix.*

A matrix A is said to be symmetric if $A' = A$. For example, the matrix

A = $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & b_2 & c_2 \\ a_3 & c_2 & c_3 \end{bmatrix}$

is symmetric.

A matrix A is said to be skew-symmetric if $A' = -A$. For example, the matrix

$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

is skew-symmetric.

Note

A visual criterion for the symmetry of a matrix would, therefore, be that it is symmetric along the main diagonal, whereas the matrix would be skew-symmetric if all the elements on the main diagonal are zeros and the elements symmetric about the main diagonal are equal in magnitude but opposite in signs.

EXERCISE 1.1

1. If a matrix has 12 elements, what are the possible orders it can have ? What if it has 7 elements ?
2. Construct a 3×4 matrix whose elements are

   (i) $a_{ij} = i + j$ (ii) $a_{ij} = i - j$

   (iii) $a_{ij} = i\,j$ (iv) $a_{ij} = \dfrac{i}{j}$
3. If $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$,

   find x, y, z, w
4. Let A, B be matrices such that $A + B$ is defined. Show that

   $(A+B)' = A' + B'$ and

   $(kA)' = kA'$, where k is a number.
5. A trust fund has Rs 30000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year Using matrix multiplication, determine how to divide Rs 30000 among the two types of bonds if the trust fund must obtain an annual total interest of :

   (a) Rs 1800 (b) Rs 2000 (c) Rs 1600.
6. Three shopkeepers A, B and C go to a store to buy stationery. A purchases 12 dozen notebooks, 5 dozen pens and 6 dozen pencils. B purchases 10 dozen notebooks, 6 dozen pens and 7 dozen pencils. C purchases 11 dozen notebooks, 13 dozen pens and 8 dozen pencils. A note book costs 40 paise, a pen costs Rs 1.25 and a pencil costs 35 paise. Use matrix multiplication to calculate each individual's bill.
7. The cooperative stores of a particular school has 10 dozen physics books, 8 dozen chemistry books and 5 dozen mathematics books. Their selling prices are Rs 8.30, Rs 3.45 and Rs 4.50 each respectively. Find the total amount the store will receive from selling all the items
8. Let A, B be matrices such that AB is defined. Show that $(AB)' = B'A'$.
9. Let A be a $m \times n$ matrix and I be the $n \times n$ identity matrix. Show that

   $AI = A$.
10. Let A be a $m \times n$ matrix and I be the $m \times m$ matrix. Show that $IA = A$.
11. Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$,

    $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

    Find each of the following :

    (i) $A + B$ (ii) $A - B$ (iii) $3A - C$

    (iv) AB (v) BA (vi) $(BC)'$

12. Add the matrices in the following :

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

(ii) $\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

(iii) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

13. Compute the indicated products :

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

(ii) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & d \end{bmatrix}$

(iii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

(vi) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

(vii) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$

14. Evaluate the following :

(i) $\left(\begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

(iii) $\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix} \right)$

15. Show that.

(i) $\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

16. Let A be a square matrix. Show that $\frac{1}{2}(A + A')$ is a symmetric matrix and $\frac{1}{2}(A - A')$ is a skew symmetric matrix.

Conclude that any square matrix can be written as sum of a symmetric matrix and a skew symmetric matrix.

17. Show that the elements on the main diagonal of a skew symmetric matrix are all zeros.

18. Prove the following by the principle of mathematical induction :

If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

for any positive integer n.

19. Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Show by mathematical induction that

$A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$

for every positive integer n.

20. Let $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I, the identity matrix of order 2. Show that

$I + A = (I - A) \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

21. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Show that

$(aI + bA)^n = a^n I + na^{n-1} bA,$

where I is the identity matrix of order 2 and n is a positive integer.

22. Let $f(x) = x^2 - 5x + 6$. Find $f(A)$ if

$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

1.4 Determinants

For every square matrix $A = [a_{ij}]$, we associate a number called determinant of A denoted by $\det A$ or $|A|$ or even $|a_{ij}|$. The notation $|a_{ij}|$ appears to be somewhat confusing as it might be confused for the absolute value of the element a_{ij} of the matrix. However, in case of matrices the notation has become standard. In this section we shall learn how to associate the number $\det A$ to the square matrix A *The matrices which are not square do not have determinants*. The determinant are very useful in solving a system of linear equations as will be illustrated in section 1.6. The determinants have also been used as a convenient way of expressing certain formulas. We shall illustrate this by expressing the formula for area of the triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3). The determinants are very useful in the study of matrices. We shall see one such use while discussing invertible matrices.

The determinant of a 1×1 matrix $[a]$ is defined to be a.

Let us now define the determinant of a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We define $|A| = ad - bc$.

We also write $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

The determinant of a 3×3 matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \text{is denoted by}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{and is defined to be}$$

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \quad \text{...(1.3)}$$

The determinant of a $n \times n$ matrix is called a determinant of order n. Before giving the definition of a determinant of order n, in general, let us consider an example of a determinant of order 3.

Example 1.13

Find the value of the determinant

$$\Delta = \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$$

By definition

$$\Delta = 1 \begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} + 3 \begin{vmatrix} 4. & 2 \\ 3 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & -1 \\ 3 & 5 \end{vmatrix}$$

$= (-2 - 10) + 3\ (8 - 6) + 2\ (20 + 3)$

$= -12 + 6 + 46 = 40.$

Going back to definition of the determinant of order 3, we say that the determinant had been expanded along the first row as in (1.3). Its expansion along the second row will be

$$-b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \qquad \text{...(1.4)}$$

It is easily verified that the value given by (1.3) is the same as given by (1.4). We can also expand it along any column. For example, expanding it along column 2, we get its value equal to

$$-a_1 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

What is the technique of finding these expansions? We shall illustrate this by defining the value of any determinant of order n given by

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \qquad \text{.. (1.5)}$$

For this purpose, first we define the minor and cofactor of the element a_{ij} appearing in the ith row and jth column of the determinant. If we supress the ith row and the jth column of the determinant, we get a determinant M_{ij} of order $(n-1)$. This determinant M_{ij} is defined to be the *minor* of the element a_{ij}. The *cofactor* C_{ij} of a_{ij} is defined to be $(-1)^{i+j} M_{ij}$.

Example 1.14

Find all minors and cofactors of the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} .$$

Solution

$$M_{11} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \qquad C_{11} = (-1)^{1+1} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \qquad C_{12} = - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \qquad C_{13} = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} \qquad C_{21} = - \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}$$

$$M_{22} = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} \qquad C_{22} = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix}$$

$$M_{23} = \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \qquad C_{23} = - \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

$$M_{31} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \qquad C_{31} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

$$M_{32} = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \qquad C_{32} = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

$$M_{33} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \qquad C_{33} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Now, we can easily define the value of the determinant Δ of order n, given by (3) :

$$\Delta = \sum_{j=1}^{n} (-1)^{i+j} \; a_{ij} \; M_{ij}$$

$$= \sum_{j=1}^{n} a_{ij} C_{ij} .$$

The above expression is obtained by expanding the determinant along ith row. If we expand it along jth column its value turns out to be

$$\sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij} .$$

The value of the determinant does not depend on along which row or column it is expanded. We are not giving a proof of this result However, let us verify it for the determinant

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

by expanding it along first row, first column and second column.

Along first row :

$$\Delta = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 b_2 c_3 - a_1 c_2 b_3 - a_2 b_1 c_3 + a_2 c_1 b_3 + a_3 b_1 c_2 - a_3 c_1 b_2$$

Along first column :

$$\Delta = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

$$= a_1 b_2 c_3 - a_1 c_2 b_3 - b_1 a_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - c_1 b_2 a_3$$

Along second Column :

$$\Delta = -a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

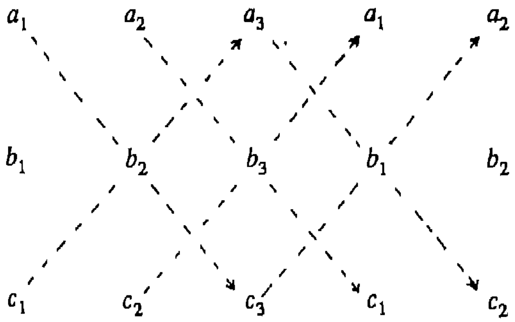
$$= -a_2 b_1 c_3 + a_2 c_1 b_3 + b_2 a_1 c_3 - b_2 c_1 a_3 - c_2 a_1 b_3 + c_2 b_1 a_3$$

It is clear that all the three expressions for Δ are equal.

We also give a fairly easy method of evaluating a determinant of order 3. Consider

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Write the three columns and repeat columns. Draw the arrows as shown in the figure below. Add the products on the downward moving arrows and subtract the product of the numbers lying on the upward moving arrow This method does not work for determinants of order greater than 3.

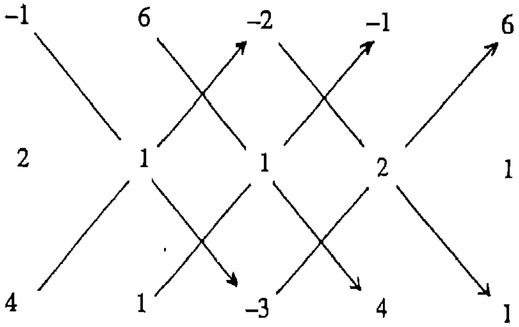


Example 1.15 .

Evaluate

$$\Delta = \begin{vmatrix} -1 & 6 & -2 \\ 2 & 1 & 1 \\ 4 & 1 & -3 \end{vmatrix}$$

We have the following diagram



Hence, $\Delta = (3 + 24 - 4) - (-8 - 1 - 36)$
$= 68$

1.5 Properties of Determinants

The following properties of determinants are true for determinants of any order. However, we shall prove them for those of order 3

These properties are often used to simplify the determinant before expanding it.

Theorem 1.4

The value of a determinant remains unchanged if its rows and columns are interchanged.

Proof : Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Then,

$$\Delta = a_1 (b_2c_3 - c_2b_3) - a_2 (b_1c_3 - c_1b_3) + a_3 (b_1c_2 - b_2c_1)$$

Let us interchange the rows and columns of Δ. Then we get the determinant

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding along first column, we get

$$\Delta_1 = a_1 (b_2c_3 - c_2 b_3) - a_2 (b_1c_3 - c_1 b_3) + a_3 (b_1c_2 - b_2 c_1)$$

Hence, $\Delta = \Delta_1$.

Remarks

(1) Since interchange of rows and columns does not change the value of the determinant, whatever we prove for rows remains true for columns.

(2) If A is a square matrix, then det A = det A'.

Theorem 1.5

If two rows or columns of a determinant are interchanged, then the sign of the determinant is changed.

Proof : Let

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Let us interchange the first and third rows. [The arguments for other interchange are similar]. The new determinant after the interchange is now

$$\Delta_1 = \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Then

$$\Delta = a_1 (b_2c_3 - c_2b_3) - a_2 (b_1c_3 - c_1b_3) + a_3 (b_1c_2 - c_1b_2).$$

Expanding Δ_1 along the third row, we get

$$\Delta_1 = a_1 (c_2b_3 - b_2c_3) - a_2 (c_1b_3 - b_1c_3) + a_3 (c_1b_2 - b_1c_2).$$

It is now clear that $\Delta_1 = -\Delta$.

Theorem 1.6

If any two rows or columns of a determinant are identical, then its value is zero.

Proof · If we interchange the identical rows of Δ, then Δ does not change. However, from theorem 1.5, it follows that Δ has changed its sign. Thus Δ = − Δ.

Therefore, Δ = 0

Theorem 1.7

If each element of a row or column of a determinant is multiplied by a constant k then its value gets multiplied by k

Proof. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let Δ_1 be the determinant obtained by multiplying the elements of the first row by k. Then

$$\Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= ka_1\,(b_2c_3 - c_2b_3) - kb_1\,(a_2c_3 - a_3c_2) + kc_1\,(a_2b_3 - a_3b_2)$$
$$= k\,[\,a_1\,(b_2c_3 - c_2b_3) - b_1\,(a_2c_3 - a_3c_2) + c_1\,(a_2b_3 - a_3b_2)\,]$$
$$= k\,\Delta.$$

If the elements of any other row or column are multiplied by k, we get the result by expanding along that row or column.

Note on the other hand that in the case of matrices kA is the matrix obtained when every entry of A has been multiplied by k and hence if A is square matrix of order n then

$$|kA| = k^n\,|A|$$

Corollary 1·1

If two rows (or columns) of a determinant are proportional, then its value is zero

Proof Let

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ kb_1 & kb_2 & kb_3 \end{vmatrix}$$

$$= k \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \text{(by theorem 1.7)}$$

$$= k\,0 \qquad \text{(by theorem 1.6)}$$
$$= 0$$

Corollary 1.2

Let $A = [a_{ij}]$ be a square matrix of order n.
Let C_{ij} denote the cofactor of the element a_{ij}.

Then,

$$\sum_{i=1}^{n} a_{ik}\ C_{jk} = \begin{cases} |A| \text{ if } i=j \\ 0 \text{ if } i \neq j \end{cases}$$

Proof : If $i=j$, the expression $\sum_{k=1}^{n} a_{ik}C_{jk}$ is

$$\sum_{k=1}^{n} a_{ik}C_{jk} = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$$

$= |A|$ as it is the expansion of the determinant along the ith row.

If $i \neq j$, consider the following determinant :

$$\begin{array}{l|cccc|} & a_{11} & a_{12} & \dots & a_{1n} \\ & a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & & \vdots \\ i\text{th row} \longrightarrow & a_{i1} & a_{i2} & \dots & a_{in} \\ & \vdots & & & \vdots \\ j\text{th row} \longrightarrow & a_{j1} & a_{j2} & \dots & a_{jn} \\ & \vdots & & & \vdots \\ & a_{n1} & a_{n2} & \dots & a_{nn} \end{array}$$

The value of this determinant is zero as it has two identical rows. However, expanding the above determinant along the jth row, we get the value of the determinant as $\sum_{k=1}^{n} a_{ik}C_{jk}$ (since $a_{jk} = a_{ik}$) and this is zero.

Hence, we conclude that if $i \neq j$, then $\sum_{k=1}^{n} a_{ik}C_{jk} = 0$.

Note

If instead of rows, we consider columns in the proof, we obtain

$$\sum_{k=1}^{n} a_{ki}C_{kj} = \begin{cases} |A| \text{ if } i=j \\ 0 \text{ if } i \neq j \end{cases}$$

Theorem 1 8

If each element of a row (or column) of a determinant is expressed as a sum of two or more terms, then the determinant can be expressed as sum of two or more determinants . Thus

$$D = \begin{vmatrix} a_1+\alpha_1 & a_2+\alpha_2 & a_3+\alpha_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Proof . Expanding the determinant on the left hand side along the first row, we get

$$\Delta = (a_1+\alpha_1)\ (b_2c_3 - c_2b_3) - (a_2+\alpha_2)\ (b_1c_3 - c_1b_3) + (a_3+\alpha_3)\ (b_1c_2 - c_1b_2)$$

$$= a_1\ (b_2c_3 - c_2b_3) - a_2\ (b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2) + \alpha_1(b_2c_3 - c_2b_3) - \alpha_2(b_1c_3 - c_1b_3) + \alpha_3(b_1c_2 - c_1b_2)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

If the elements of any other row or column are expressed as the sum of two terms, we get the same result by expanding it along that row or column.

Theorem 1.9

If to any row or column of a determinant, a multiple of another row or column is added, the value of the determinant remains the same.

Proof.

$$\text{Let } \Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Suppose
$$\begin{vmatrix} a_1+kc_1 & a_2+kc_2 & a_3+kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

that is, Δ_1 is the determinant obtained from Δ where k-times the third row of Δ is added to its first row. (The argument in case of other rows or columns is similar). Now

$$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + k \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

by Theorems 1.7 and 1 8

The second determinant on the right hand side vanishes by Theorem 1.6. So we get $\Delta_1 = \Delta$.

Example 1 16

If ω is one of the imaginary cube roots of unity, find the value of

$$\Delta = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

Adding the second and third rows to the first row of Δ we get

$$\Delta = \begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} \qquad (\text{because } 1+\omega+\omega^2=0)$$

$$= 0. \qquad (\text{expanding along the first row})$$

Example 1 17

Evaluate

$$\begin{vmatrix} 219 & 117 & 345 \\ 19 & 9 & 34 \\ 7 & 3 & 5 \end{vmatrix}$$

Solution

$$\Delta = \begin{vmatrix} 219 & 117 & 345 \\ 19 & 9 & 34 \\ 7 & 3 & 5 \end{vmatrix}$$ (Substract 10 times the second row from the first row, that is, $R_1 - 10\,R_2$)

$$= \begin{vmatrix} 29 & 27 & 5 \\ 19 & 9 & 34 \\ 7 & 3 & 5 \end{vmatrix}$$ ($R_1 - R_3$ and $R_2 - 3R_3$)

$$= \begin{vmatrix} 22 & 24 & 0 \\ -2 & 0 & 19 \\ 7 & 3 & 5 \end{vmatrix}$$ (Take 2 common out in the first row)

$$= 2\begin{vmatrix} 11 & 12 & 0 \\ -2 & 0 & 19 \\ 7 & 3 & 5 \end{vmatrix}$$ (Take 3 common out in the second column)

$$= 6\begin{vmatrix} 11 & 4 & 0 \\ -2 & 0 & 19 \\ 7 & 1 & 5 \end{vmatrix}$$ (Expand as entries are now small)

$$= 6[-11 \times 19 - 4(-10 - 133)]$$
$$= 6[-209 + 572] = 6 \times 363 = 2178$$

Example 1 18

Show that

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6b+3b & 10a+6b+3c \end{vmatrix} = a^3.$$

Solution

$$\Delta = \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + \begin{vmatrix} a & b & a+b+c \\ 2a & 2b & 4a+3b+2c \\ 3a & 3b & 10a+6b+3c \end{vmatrix}$$

(The second determinant is zero, because I and II columns are proportional)

$$= \begin{vmatrix} a & a & a \\ 2a & 3a & 4a \\ 3a & 6a & 10a \end{vmatrix} + \begin{vmatrix} a & a & b \\ 2a & 3a & 3b \\ 3a & 6a & 6b \end{vmatrix} + \begin{vmatrix} a & a & c \\ 2a & 3a & 2c \\ 3a & 6a & 3c \end{vmatrix}$$

(The second and third determinants are zeros)

$$= a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$ (Applying $C_2 - C_1$ and $C_3 - C_1$)

$$= a^3 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix} = a^3 .1\ (7-6) = a^3.$$

Note

$C_2 - C_1$ means subtracting first column from second column. Similar statements regarding rows and columns have their obvious natural meanings.

EXERCISE 1.2

1. Evaluate the following determinants :

(i) $\begin{vmatrix} 3 & 4 \\ 9 & -7 \end{vmatrix}$ (ii) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

(iii) $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$ (iv) $\begin{vmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{5} \end{vmatrix}$

2. Write the minors and cofactors of each element of the first column of the following determinants and evaluate the determinant in each case:

(i) $\begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

(iii) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ (iv) $\begin{vmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{vmatrix}$

3. Evaluate the following determinants:

(i) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ (ii) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

(iii) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & -1 & 0 & 1 \\ -3 & 0 & 1 & -2 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 5 & 0 \end{vmatrix}$

(v) $\begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$ (vi) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

Prove the following identities :

4 $$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

5. $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)\ (c-a)\ (a-b).$$

6 $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-c)\ (c-a)\ (a-b)\ (a+b+c)$$

7. $$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (y-z)\ (z-x)\ (x-y)\ (yz+zx+xy).$$

8. $$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4\ (a+b)\ (b+c)\ (c+a).$$

9. $$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\ (a+b+c)^3.$$

10 $$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = bc+ca+ab+abc.$$

11. $$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4\ a^2\ b^2\ c^2.$$

1.6. Application of Determinants

In this section we shall discuss the use of determinants in finding the area of a triangle and in solving a system of linear equations.

Area of a Triangle

In the previous class, you learnt that the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by the expression :

$$\frac{1}{2}\left[\ x_1(y_2-y_3)-y_1(x_2-x_3)+x_2y_3-x_3y_2\ \right]$$

As you can see, this expression is the expansion of the determinant

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

So the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Since the area has to be a positive quantity, we always take the absolute value of the above determinant for the area.

Example 1.19

Find the area of the triangle with vertices (-2, 4), (2, -6) and (5, 4)

Solution

The area Δ of the triangle

$$= \frac{1}{2}\begin{vmatrix} -2 & 4 & 1 \\ 2 & -6 & 1 \\ 5 & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[(12-8)-(-8-20)+(8+30)] \qquad \text{(by expanding along third column)}$$

$$= \frac{1}{2}[4+28+38]=35.$$

Solution of Systems of Linear equations by Determinants

Let us consider the system:

$$a_1x+b_1y=c_1$$
$$a_2x+b_2y=c_2$$

Solving these equations, we get

$$x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}, \quad y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1},$$

Writing determinants, we can express the solution as

$$x=\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y=\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Observation

We note that the denominator in the values of x and y is really the determinant of the coefficients and the numerators are the determinants obtained by replacing the coefficients of x and y respectively by the constant terms. We shall elaborate on this point while dealing with the general case.

Example 1.20

Using determinants, solve the following system of equations:

$$2x-4y=-3$$
$$4x+2y=9$$

Solution

$$x=\frac{\begin{vmatrix} -3 & -4 \\ 9 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -4 \\ 4 & 2 \end{vmatrix}}, \qquad y=\frac{\begin{vmatrix} 2 & -3 \\ 4 & 9 \end{vmatrix}}{\begin{vmatrix} 2 & -4 \\ 4 & 2 \end{vmatrix}}$$

Hence, $x=\frac{30}{20}$, $y=\frac{30}{20}$

or $x=\frac{3}{2}$, $y=\frac{3}{2}$

Let us now consider the following system of equations :

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

Let

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Then

$$xD=\begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}.$$

Multiplying the second column by y and the third column by z and adding these to the first column, we get

$$xD=\begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix}$$

or $$xD = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = D_1, \text{ say.}$$

Let $$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Now, as before, we can see that $yD = D_2$ and $zD = D_3$.

Thus $x = \dfrac{D_1}{D}$, $y = \dfrac{D_2}{D}$, $z = \dfrac{D_3}{D}$, provided $D \neq 0$.

The method which we used for solving a system of three equations can be used in exactly the same way to solve a system of n equations in n unknowns. Below we state the theorem for the general case. The theorem is known as Cramer's Rule after the Swiss mathematician Gabriel Cramer (1704 – 1752).

Theorem 1.10

Consider the system of n linear equations in n unknowns given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

Let

$$D = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the jth column by

$$\begin{vmatrix} b_1 \\ \\ b_n \end{vmatrix}$$

Then, if $D \neq 0$, we have $x_1 = \dfrac{D_1}{D}$, $x_2 = \dfrac{D_2}{D}$, ..., $x_n = \dfrac{D_n}{D}$.

Example 1.21

Solve the following system of equations :

$$\begin{aligned} 2x + 5y - z &= 9 \\ 3x - 3y + 2z &= 7 \\ 2x - 4y + 3z &= 1 \end{aligned}$$

Solution

$$D = \begin{vmatrix} 2 & 5 & -1 \\ 3 & -3 & 2 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 2(-9+8) - 5(9-4) - (-12+6)$$

$$= -2 - 25 + 6 = -21$$

$$D_1 = \begin{vmatrix} 9 & 5 & -1 \\ 7 & -3 & 2 \\ 1 & -4 & 3 \end{vmatrix}$$

$$= 9(-9+8) - 5(21-2) - (-28+3)$$

$$= -9 - 95 + 25 = -79$$

$$D_2 = \begin{vmatrix} 2 & 9 & -1 \\ 3 & 7 & 2 \\ 2 & 1 & 3 \end{vmatrix} = 2(21-2) - 9(9-4) - (3-14) = 38 - 45 + 11 = 4$$

$$D_3 = \begin{vmatrix} 2 & 5 & 9 \\ 3 & -3 & 7 \\ 2 & -4 & 1 \end{vmatrix} = 2(-3+28) - 5(3-14) + 9(-12+6) = 50 + 55 - 54 = 51$$

Hence, $x = \dfrac{79}{21}$, $y = \dfrac{-4}{21}$, $z = \dfrac{-17}{7}$

Example 1 22

Solve

$$\begin{aligned} x + y + z + w &= 2 \\ x - 2y + 2z + 2w &= -6 \\ 2x + y - 2z + 2w &= -5 \\ 3x - y + 3z - 3w &= -3 \end{aligned}$$

Solution

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 2 & 2 \\ 2 & 1 & -2 & 2 \\ 3 & -1 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ 2 & -1 & -4 & 0 \\ 3 & -4 & 0 & -6 \end{vmatrix}$$

$[C_2 - C_1,\ C_3 - C_1$ and $C_4 - C_1]$

$$= \begin{vmatrix} -3 & 1 & 1 \\ -1 & -4 & 0 \\ -4 & 0 & -6 \end{vmatrix} = -3 \times 24 - 6 - 16 = -94$$

$$D_1 = \begin{vmatrix} 2 & 1 & 1 & 1 \\ -6 & -2 & 2 & 2 \\ -5 & 1 & -2 & 2 \\ -3 & -1 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 & 1 \\ -6 & -4 & 0 & 2 \\ -5 & -1 & -4 & 2 \\ -3 & 2 & 6 & -3 \end{vmatrix} \quad \begin{bmatrix} C_2 - C_4 \\ C_3 - C_4 \end{bmatrix}$$

$$D_1 = 2 \begin{vmatrix} -4 & 0 & 2 \\ -1 & -4 & 2 \\ 2 & 6 & -3 \end{vmatrix} - \begin{vmatrix} -6 & -4 & 0 \\ -5 & -1 & -4 \\ -3 & 2 & 6 \end{vmatrix}$$

$$= 2\ [\,-4\ (12 - 12) + 2\ (-6 + 8)\,] - [\,-6\ (-6 + 8) + 4\ (-30 - 12)\,]$$

$$= \quad 8 - [\,-12 - 42 \times 4\,] = 188$$

$$D_2 = \begin{vmatrix} 1 & 2 & 1 & 1 \\ 1 & -6 & 2 & 2 \\ 2 & -5 & -2 & 2 \\ 3 & -3 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & -7 & 1 & 1 \\ 2 & -7 & -4 & 0 \\ 3 & -6 & 0 & -6 \end{vmatrix} \qquad \begin{bmatrix} C_3 - C_1 \\ C_4 - C_1 \end{bmatrix}$$

$$= \begin{vmatrix} -7 & 1 & 1 \\ -7 & -4 & 0 \\ -6 & 0 & -6 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ 2 & -4 & 0 \\ 3 & 0 & -6 \end{vmatrix}$$

$$= \quad -7 \times 24 - 42 - 24 - [\,24 + 12 + 12\,]$$

$$= \quad -234 - 48 = -282$$

$$D_3 = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 1 & -2 & -6 & 2 \\ 2 & 1 & -5 & 2 \\ 3 & -1 & -3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & -3 & -7 & 1 \\ 2 & -1 & -7 & 0 \\ 3 & -4 & -6 & -6 \end{vmatrix} \qquad \begin{bmatrix} C_2 - C_1, C_3 - C_1 \\ C_4 - C_1 \end{bmatrix}$$

$$= \begin{vmatrix} -3 & -7 & 1 \\ -1 & -7 & 0 \\ -4 & -6 & -6 \end{vmatrix} + \begin{vmatrix} 1 & -3 & 1 \\ 2 & -1 & 0 \\ 3 & -4 & -6 \end{vmatrix}$$

$$= \quad -3 \times 42 + 7 \times 6 + (6 - 28) + 6 + 3\ (-12) + (-8 + 3)$$

$$= \quad -126 + 42 - 22 + 6 - 36 - 5 = -\ 189 + 48 = -\ 141$$

$$D_4 = \begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & -2 & 2 & -6 \\ 2 & 1 & -2 & -5 \\ 3 & -1 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 2 \\ 1 & -3 & 1 & -6 \\ 2 & -1 & -4 & -5 \\ 3 & -4 & 0 & -3 \end{vmatrix} \qquad [C_2 - C_1, C_3 - C_1]$$

$$= \begin{vmatrix} -3 & 1 & -6 \\ -1 & -4 & -5 \\ -4 & 0 & -3 \end{vmatrix} -2 \begin{vmatrix} 1 & -3 & 1 \\ 2 & -1 & -4 \\ 3 & -4 & 0 \end{vmatrix}$$

$$= \quad -\ 36 - (3 - 20) - 6\ (-\ 16) - 2\ [\,-\ 16 + 36 - 5\,]$$

$$= \quad -\ 36 + 17 + 96 - 30 = 47.$$

Hence, $x = \frac{D_1}{D} = \frac{188}{-94} = -2$, Similarly, $y = 3, z = \frac{3}{2}$ and $w = \frac{-1}{2}$.

Note

1. For the system in Theorem 1.10 if $b_1 = b_2 = ... = b_n = 0$, then each $D_i = 0$ and if $D \neq 0$, the system has only the trivial solution $x_1 = x_2 = ... = x_n = 0$.

2. Cramer's rule does not apply if $D = 0$.

3. While discussing the system of three equations in three unknowns, we saw that $x D = D_1$, $y D = D_2$ and $z D = D_3$ for any x, y, z satisfying the system of equations. Thus if $D = 0$ and some $-D_i \neq 0$, then the system has no solution. Same remark holds good in the general case.

4. If $D = 0$ (and $D_1 = D_2 = D_3 = 0$), we may still use the method of determinants as illustrated by the following example.

Example 1.23

Solve the following system of equations :

$$x - y + 3z = 6$$
$$x + 3y - 3z = -4$$
$$5x + 3y + 3z = 10$$

Solution

$$D = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 3 & -3 \\ 5 & 3 & 3 \end{vmatrix} = 3 \begin{vmatrix} 1 & -1 & 1 \\ 1 & 3 & -1 \\ 5 & 3 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 4 & -2 \\ 5 & 8 & -4 \end{vmatrix} \quad [-C_1 + C_2, C_3 - C_1]$$

= 0, as 2nd and 3rd columns are scalar multiple of each other.

Considering first two equations, we have

$$x - y = 6-3z$$
$$x + 3y = 3z-4$$

Hence,

$$x = \frac{\begin{vmatrix} 6-3z & -1 \\ 3z-4 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}} \quad \text{and } y = \frac{\begin{vmatrix} 1 & 6-3z \\ 1 & 3z-4 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}}$$

$$\text{or } x = \frac{18-9z + 3z-4}{4} = \frac{14-6z}{4} = \frac{7-3z}{2}$$

$$\text{and } y = \frac{3z-4-6+3z}{4} = \frac{6z-10}{4} = \frac{3z-5}{2}$$

We can give arbitrary values to z and find the corresponding values of x, y Thus we have infinite number of solutions satisfying the first two equations These are also seen to satisfy the third equation

EXERCISE 1.3

1 Find the area of the triangle with vertices at the points given in each of the following problems :

(a) (0, 0), (6, 0), (4, 3)
(b) (3, 8), (−4, 2), (5, −1)
(c) (2, 7), (1, 1), (10, 8)
(d) (−2, −3), (3, 2), (−1, −8)

2. Using determinants, solve the following systems of equations :

(i) $5x + 7y = -2$
$4x + 6y = -3$

(ii) $3x + ay = 4$
$2x + ay = 2$

(iii) $2x + 3y + z = 9$
$4x + y = 7$
$x - 3y - 7z = 6$

(iv) $6x + y - 3z = 5$
$x + 3y - 2z = 5$
$2x + y + 4z = 8$

(v) $3x - 5z = -1$
$2x + 7y = 6$
$x + y + z = 5$

(vi) $2x - 3y - z = 0$
$x + 3y - 2z = 0$
$x - 3y = 0$

(vii) $x - y = 1$
$x + z = -6$
$x + y - 2z = 3$

(viii) $2x - 3z + w = 1$
$x - y + 2w = 1$
$-3y + z + w = 1$
$x + y + z = 1$

1.7. Adjoint and Inverse of a Matrix and its applications in Solving Linear Equations

Definition

A nonzero square matrix A is said to be invertible if there exists a square matrix B of order n such that

(1) $AB = BA = I$,

where I is that identity matrix of order n, B is called an inverse of A and A is called an inverse of B An invertible matrix has a unique inverse. This follows from the following

Theorem 1 11

If A, B, C are square matrices of order n such that
$AB = BA = I$ and $AC = CA = I$,
then $B = C$

Proof. $B = B\,I = B\,(A\,C) = (B\,A)\,C = I\,C = C$
(give reasons for each of the above steps)
Hence, if A is invertible, it has a unique inverse, which we denote by A^{-1} Thus, we have $A\,A^{-1} = A^{-1}\,A = I$.

Remark

In order that matrices A, B satisfy $A\,B = B\,A = I$ for the identity matrix of order n, it is necessary that A, B be both square matrices of order n. This can be seen as follows :
Suppose A is if order $p \times q$ and B is of order $r \times s$. In order that $A\,B$ and $B\,A$ be defined, it is necessary that $q = r$ and $s = p$. Then AB is a $p \times s$ matrix and $B\,A$ is a $r \times q$ matrix. Since both are equal to I (order n), we must have $p = s = r = q = n$.
Hence, it follows that it is meaningful to talk about an inverse of a matrix only for square matrices.

Let us now consider the problem of finding the inverse of an invertible matrix. We begin with a square matrix of order 2. Let

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ We want to find a matrix

$B = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$ such that $AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now,

$$AB = \begin{bmatrix} ax_1 + bx_2 & ay_1 + by_2 \\ cx_1 + dx_2 & cy_1 + dy_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence,
$$\begin{aligned} ax_1 + bx_2 &= 1 \\ cx_1 + dx_2 &= 0 \\ ay_1 + by_2 &= 0 \\ cy_1 + dy_2 &= 1 \end{aligned} \qquad \ldots(1.6)$$

Suppose $|A| = ad - bc \neq 0$:

Solving the above equations

$$x_1 = \frac{d}{|A|},\ x_2 = \frac{-c}{|A|},\ y_1 = \frac{-b}{|A|},\ y_2 = \frac{a}{|A|}$$

Hence, *if* $B = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, then

$AB = I$.

$$\text{Also, } BA = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{|A|}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, $AB = BA = I$

Thus, $B = A^{-1}$.

Thus, we see that if $|A| \neq 0$, then A is invertible.

It can also be verified that if $|A| = 0$, then A cannot be invertible, because solving the system of equations (1 6), we get

$x_1 |A| = d_1 |A| x_2 = -c, y_1 |A| = -b$ and $y_2 |A| = a$ Hence, if $|A| = 0$ and $B = \begin{bmatrix} x_1 & y_1 \\ x_1 & y_1 \end{bmatrix}$ exists satisfying $AB = BA = BA = I$, then $a = b = c = d = 0$. But then $A = O$ and it can not have an inverse. Hence, we see that for a square matrix A of order 2, A is invertible if and only if $|A| \neq 0$.

Definition

A square matrix is said to be *non-singular* if $|A| \neq 0$, otherwise A is said to be ***singular***. Thus for a square matrix A of order 2, A is invertible if and only if A is non-singular. What we have seen for square matrices of order 2 is true for square matrices of any order. This we state as a theorem below without proof.

Theorem 1 12

A square matrix A is invertible if and only if A is non-singular.

Example 1 24

The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular if $ad - bc \neq 0$ The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is non-singular. Why ?

Example 1 25

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ 2 & -3 & 1 \end{bmatrix}$$

Then,

$$|A| = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ 2 & -3 & 1 \end{vmatrix} = (2+18) - (2-12) + 3(-6-4) = 20 + 10 - 30 = 0.$$

Hence, A is singular.

In view of Theorem 1.12, it is easy to decide if a given square matrix A is invertible. The problem then remains of finding A^{-1} if A is invertible. We discuss the method for square matrices of order 3 and then state the general result But first we need a definition.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdot\cdot & a_{1n} \\ a_{21} & a_{22} & \cdot\cdot & a_{2n} \\ \cdot & & & \cdot \\ a_{n1} & a_{n2} & \cdot & a_{nn} \end{bmatrix}$$

Let A_{ij} = Cofactor of the element a_{ij}

Definition

The *adjoint* of A is defined to be the transpose of the matrix $[A_{ij}]$ and is denoted by adj A

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

then the adjoint of A is given by

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Theorem 1.13

If A is a square matrix $[a_{ij}]$ and $|A| \neq 0$, then $A^{-1} = \dfrac{1}{|A|} \times$ adjoint of A.

Proof : We demonstrate the theorem for a square matrix A of order 3. The proof of the general case is similar and hence omitted.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

adj $A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Now, the (i, j)th element of A.adj A

$= a_{i1}A_{j1} + a_{i2}A_{j2} + a_{i3}A_{j3}$

$= |A|$ if $i = j$

$= 0$ if $i \neq j$.

by Corollary 1.2

Hence,

$$A \quad \frac{\text{adj } A}{|A|} = I.$$

Similarly, we can see that

$$\frac{\text{adj } A}{|A|} \cdot A = I$$

by using the note following corollary 1.2.

This proves that $A^{-1} = \dfrac{1}{|A|}$ adj A.

Example 1.26

Let $A = \begin{bmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{bmatrix}$

Show that A is invertible. Find adj A and A^{-1}.

Solution

$$\begin{aligned} |A| &= \begin{vmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{vmatrix} \\ &= 4\,(1-11) + 6\,(1+4) + (-11-4) \\ &= -40 + 30 - 15 = -25\,. \end{aligned}$$

Hence, $|A| \neq 0$ and A is invertible

Now,

$A_{11} = -10 \quad A_{12} = -5 \quad A_{13} = -15$
$A_{21} = 5 \quad A_{22} = 0 \quad A_{23} = -20$
$A_{31} = -5 \quad A_{32} = -5 \quad A_{33} = -10$

$$\text{adj}\, A = \begin{bmatrix} -10 & 5 & -5 \\ -5 & 0 & -5 \\ -15 & -20 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\,\text{adj}\, A = \begin{bmatrix} \frac{2}{5} & \frac{-1}{5} & \frac{1}{5} \\ \frac{1}{5} & 0 & \frac{1}{5} \\ \frac{3}{5} & \frac{4}{5} & \frac{2}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & 4 & 2 \end{bmatrix}$$

Let us verify that $A\,A^{-1} = I$.

$$\begin{aligned} A\,A^{-1} &= \frac{1}{5}\begin{bmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & 4 & 2 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 4\times 2-6+3 & 0 & 0 \\ 0 & 5 & 0 \\ -8+11-3 & 0 & -4+11-2 \end{bmatrix}. \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= I. \end{aligned}$$

Linear Equations in Matrix Notation

Consider the following system of linear equations :

$$a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \,.\,\,. + a_{2n}x_n = b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Let A be the *coefficient matrix*, that is, let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \,.\,\,. & a_{2n} \\ \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Suppose $X = \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \\ b_m \end{bmatrix}$

Then the system of equations given above can be written in the matrix notation as $AX = B$

Note

We shall be concerned with n equations in n unknowns so that $m = n$ and A is a square matrix. Discussion of genereal case is beyond the scope of our book.

The system of equations given in above is said to be *consistent* if the system has at least one solution and the system is said to be *inconsistent* if it is not consistent.

Example 1 27

The system

$x + y = 5$

$2x + 2y = 7$

is an inconsistent system of equations, because for no values of x and y both the equations are satisfied.

Example 1 28

The system

$x + y = 5$

$2x + 3y = 7$

is consistent This system of equations can be written as

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

Then

$$|A| = 1 \text{ and } A^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Hence, $x = 8$ and $y = -3$.

In this case the system has a unique solution

From the above example, we feel that if there is a system of n linear equations in n unknowns given by $A X = B$ and A is non-singular, then the system has a unique solution. In fact, in this case it is easy to see that we have $X = A^{-1}B$.

Thus in this case the system can be solved by finding the inverse of the coefficient matrix.

Example 1 29

Solve the following system of linear equations ·

$$2x - 3y + 3z = 1$$
$$2x + 2y + 3z = 2$$
$$3x - 2y + 2z = 3$$

Solution

$$A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$$

$$\begin{aligned} |A| &= 2(4 + 6) + 3(4 - 9) + 3(-4 - 6) \\ &= 20 - 15 - 30 = -25 \end{aligned}$$

$A_{11} = 10$, $A_{12} = 5$, $A_{13} = -10$
$A_{21} = 0$, $A_{22} = -5$, $A_{23} = -5$
$A_{31} = -15$, $A_{32} = 0$, $A_{33} = 10$.

Hence,

$$A^{-1} = -\frac{1}{25} \begin{bmatrix} 10 & 0 & -15 \\ 5 & -5 & 0 \\ -10 & -5 & 10 \end{bmatrix}$$

$$= -\frac{1}{5} \begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \frac{-1}{5}\begin{bmatrix} 2-9 \\ 1-2 \\ -2-2+6 \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -7 \\ -1 \\ 2 \end{bmatrix}.$$

Hence, $x = \frac{7}{5}$, $y = \frac{1}{5}$, $z = \frac{-2}{5}$.

In the above examples, we saw systems of equations which were either inconsistent or, if consistent, with exactly one solution.

Example 1.30

Let us now consider the system

$$x + y = 1$$
$$3x + 3y = 3$$

In this case the coefficient matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$.

Since $|A| = 0$, A^{-1} does not exist. However, this does not mean that the system has no solution. In fact, in this case the system has infinite number of solutions. For any reall number b, $x = 1-b$ and $y = b$ is a solution of the system

Criterion of Consistency

We mention the following criterion (without proof) for the consistency or inconsistency of a system of linear equations given by $AX = B$, where A is a square matrix.

(i) If $|A| \neq 0$, then the system is *consistent* and has a unique solution

(ii) If $|A| = 0$ and $(\text{adj } A)B = O$, then the system is consistent and has *infinitely many solutions*

(iii) If $|A| = 0$ and $(\text{adj } A)\, B \neq O$, then the system is *inconsistent*

System of homogeneous Linear Equations

A linear equation is said to homogeneous if the constant term is zero. The equations

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

constitute a system of homogeneous linear equations. If

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

then the system of equations can be written as

$$A\,X = O.$$

Note that $x = y = z = 0$ always satisfy the equations. Hence, the system is always consistent. The solution $x = y = z = 0$ is called the *trivial solution.*

In this case the interest lies in whether the system has non-trivial solutions. Suppose A is non-singular. Then A^{-1} exists and we get

$$A^{-1}(A\,X) = O$$

$$\text{or} \quad X = O.$$

Hence, if A is non-singular, the system has only trivial solution. Thus in order that the system $A\,X = O$ has non-trivial solution, it is necessary that $|A| = 0$. In fact, we have the following

Theorem 1.14

A system of n homogeneous linear equations in n unknowns has non-trivial solutions if and only if the coefficient matrix is singular. This theorem is proved in advance courses and we do not include its proof. However, we illustrate a method for finding non-trivial solutions by an example.

Example 1.31

Find non trivial solution of the system

$$x + y + z = 0$$

$$x - y - 5z = 0$$

$$x + 2y + 4z = 0$$

Solution

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -5 \\ 1 & 2 & 4 \end{bmatrix}$$

It is easily seen that A is singular, that is, $|A| = 0$.

Considering first two equations of the system, we have

$$x + y = -z$$

$$x - y = 5z.$$

Solving, we get $x = 2z$, $y = -3z$.

Now for any number k, we let

$$x = 2k,\ y = -3k,\ z = k.$$

Then these values of x, y, z satisfy the first two equations. It can be seen that these also satisfy the third equation.

Note

In view of the criterion of consistency mentioned earlier, we have the following regarding the solution of homogeneous linear equations given by $A\,X = O$, where A is a square matrix.

(i) If $|A| \neq 0$, then the system has only *trivial solution*

(ii) If $|A| = 0$, then the system has *infinitely many solutions*
Note that if $|A| = 0$, then $(\text{adj } A)B = O$ as $B = O$.

EXERCISE 1.4

1. Find the adjoint of the following matrices :

(i) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

2. Find the inverse of the following matrices :

(i) $\begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

3. Which of the following equations are consistent and if consistent, find the solutions

(i) $2x + 3y = 5$
$3y + 2y = 2$

(ii) $x + 2y = 5$
$3x + 6y = 15$

(ii) $2x - y = 5$
$4x - 2y = 7$

(iv) $2x - 3y = 5$
$x + y = 7$

(v) $3x - y + 2z = 3$
$2x + y + 3z = 5$
$x - 2y - z = 1$

(vi) $x - y + z = 3$
$2x + y - z = 2$
$-x - 2y + 2z = 1$

(vii) $x + y - 5z = 26$
$x + 2y + z = -4$
$x + 3y + 6z = -29$

(viii) $2x + 3y - z = -15$
$3x + 5y + 2z = 0$
$x + 3y + 3z = 11$

4. Solve the following system of equations by matrix method :

(i) $3x - 2y = 7$
$5x + 3y = 1$

(ii) $4x + 2y = 3$
$3x - 4y = 5$

(iii) $2x + 3y = -1$
$x + 2y = 2$

(iv) $5x - 7y = 2$
$7x - 5y = 3$

(v) $2x + 3y + 3z = 5$
$x - 2y + z = -4$
$3x - y - 2z = 3$

(vi) $4x + 2y + 3z = 2$
$x + y + z = 1$
$3x + y - 2z = 5$

(vii) $5x - y + z = 4$
$3x + 2y - 5z = 2$
$x + 3y - 2z = 5$

(viii) $x - y + 2z = 7$
$3x + 4y - 5z = -5$
$2x - y + 3z = 12$

5. Solve the following system of homogeneous equations :

(i) $2x + 3y - z = 0$
$x - y - 2z = 0$
$3x + y + 3z = 0$

(ii) $3x + y - 2z = 0$
$x + y + z = 0$
$x - 2y + z = 0$

(iii) $x + y - 2z = 0$
$2x + y - 3z = 0$
$5x + 4y - 9z = 0$

(iv) $2x - 7y - 6z = 0$
$3x + 5y - 2z = 0$
$4x - 2y - 7z = 0$

(v) $x + y - z = 0$
$x - 2y + z = 0$
$3x + 6y - 5z = 0$

(iv) $3x + 2y + 7z = 0$
$4x - 3y - 2z = 0$
$5x + 9y + 23z = 0$

6. Let $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$.

Verify that $(A\,B)^{-1} = B^{-1}\,A^{-1}$

7 Show that if A and B are invertible square matrices of the same order, then $A\,B$ is also invertible and $(A\,B)^{-1} = B^{-1}\,A^{-1}$

CHAPTER 2

FUNCTIONS, LIMITS AND CONTINUITY

2.1 Real Functions

IN THE FIRST three sections of this chapter, we recall some concepts that you already know. These are . functions, domain, range, graphs of real-valued functions, composition of functions, etc.

Function . A function f from a set S to a set T associates, to each element x in S, a unique element $f(x)$ in T. We say that S is the domain of f and T is the codomain of f. We use the notation $f \cdot S \to T$

If to each x in S, more than one element in T are associated, we do not have a function. (Though sometimes it is called a multivalued-function, we agree not to call it a function.)

Real Function In this course, we do not require general sets S and T. We take both of them to be subsets of **R**. Here **R** denotes the set of all real numbers. We call such functions, whose values are real numbers, as real-valued functions, or in short as real functions

Remark

In general, a function f . $\mathbf{R} \to \mathbf{R}$ need not be given by a formula However, in this course, most of the functions that we consider, are given in such a form or rule that is easily described.

Examples of Real Functions : Now we give plenty of examples. These examples will be frequently met with throughout calculus.

Value of a function . Let x be an element in the domain S of a function f The element in T that is associated to x by f is denoted by $f(x)$ It is called the value of f at x. We also say that f takes the value of $f(x)$ at x.

Range · The range of f is the set of all values taken by f It is a subset of the codomain T. It may or may not be the whole of T. In the case of real functions given by a formula $f(x)$, unless the domain is explicitly stated, we shall consider the set of all real numbers x for which $f(x)$ is defined, as the domain of f.

(1) Constant Functions Let α be a fixed real number. Consider the function that associates to each real number x, this fixed number α. Then this function is called a *constant function* from **R** to **R**. Sometimes we denote this constant function either by

$$f(x) = \alpha \text{ for all } x \text{ in } \mathbf{R}$$

or more briefly, by $f(x) = \alpha$ or even more briefly by the constant function α.

There are as many constant functions as there are real numbers.
The domain of the constant function α is evidently **R** and the range is $\{\alpha\}$.

(2) Identity Function The function that associates to each real number x the same number x, is called the *identity function* from **R** to **R**. We denote this function either by writing

$$f(x) = x \text{ for all } x \text{ in } \mathbf{R}$$

or more briefly, by $f(x) = x$ or even more briefly as the function x.
The domain of the identity function is evidently **R** and its range is also **R**.

(3) Polynomial Functions . Consider the polynomial expression $2x^3 - 5x^2 + 2x - 7$. For each real number a, this associates the number $2a^3 - 5a^2 + 2a - 7$, got by substituting a for x. We say that this is a *polynomial function*. We write $p(x) = 2x^3 - 5x^2 + 2x - 7$. Similarly,

$$f(x) = x^4 + \sqrt{2}\,x,$$

$$g(x) = \tfrac{1}{2} + x - x^3$$

etc. are also polynomial functions.

But $x^{\frac{2}{3}} + 2x$ is not a polynomial.
The domain of the polynomial function is **R**.

(4) Reciprocal Function : If a is a non-zero real number, $\frac{1}{a}$ is called *the reciprocal of a*. The reciprocal of zero is not defined. The function that associates to each non-zero real number, its reciprocal, is denoted by $\frac{1}{x}$. Unlike the previous three examples, the domain of this function is not **R**. Here the domain is $\mathbf{R} - \{0\}$.

(5) Exponential Function · You have seen in the previous class that the series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

has the sum e^x

The function that associates this number e^x to each real number x, is called the *exponential function*, and is denoted merely by e^x. In later sections, we need the following properties of this function.

$e^{x+y} = e^x \cdot e^y$ for all x, y in **R**.

e^x is always positive.

$$e^{-x} = \frac{1}{e^x}$$

The domain of the exponential function e^x is **R**.

(6) Logarithmic Function : You have studied that $\log x$ is that number y such that $e^y = x$.

Here e is the sum of $1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \;$;

$\log x$ is defined only when x is positive. The function that associates $\log x$ to x is called the *logarithmic function.* Its domain is the set of positive real numbers.

The following properties will be used later :

$\log 1 = 0$
$\log (xy) = \log x + \log y$
$e^{\log x} = x.$

(7) Rational Functions . Consider an expression, that is, the quotient of two polynomial expressions For example,

$$\frac{x^3 - 5x + 3}{x^2 - 1}.$$

This gives the function that associates to each real number a, the value $\frac{a^3 - 5a + 3}{a^2 - 1}$ with the exception of those a, for which the denominator $(a^2 - 1)$ becomes zero. In this example, the domain of the function is $\mathbf{R} - \{-1, 1\}$. Similarly, $\frac{1}{x-1}$, $\frac{x+1}{x^3}$, etc. are also rational functions. The domain of a rational function is usually of the form R–a suitable finite set. The domain may be different for different rational functions.

(8) Modulus Function : For each real number x, let $|x|$ denote the absolute value of x. This is,

$$|x| = \begin{cases} x \text{ if } x \text{ is } \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

The domain of $|x|$ is **R**.

(9) Trigonometric Functions : The function that associates the number $\sin x$ to each number x, is called the sine function. Here x is the radian-measure of an angle. In Trigonometry in the last class, you have studied that the following properties hold :

The value of $\sin x$ always lies between -1 and 1.

$\sin (x + 2\pi) = \sin x$ for all x in **R**.

$\sin 0 = 0$; $\sin \frac{\pi}{6} = \frac{1}{2}$; $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$; $\sin \frac{\pi}{2} = 1$;

$\sin (x + \pi) = -\sin x$ for all x in **R**.

Similarly, we have other trigonometric functions $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$.

You must note that their domains are not the same.

For the sine and cosine functions, the domain is **R**.

For the function tan x, the domain is $\mathbf{R} - \{(2n+1)\frac{\pi}{2} : n \text{ is an integer}\}$.

Many formulas from Trigonometry, such as $\sin(\frac{\pi}{2} + x) = \cos x$, $\sin^2 x + \text{cox}^2 x = 1$, etc. will be used in the following sections.

(10) *Square root functions* : If x is a positive real number, we know that there are two square roots for x, of which one is positive. If we associate only the positive square root, then we get a function. We denote this function by $\sqrt{x}$. Its domain is the set of all non-negative real numbers.

(11) *The greatest integer function* : If x is a real number, we denote by $[x]$, the greatest integer not exceeding x. For example, $[\frac{5}{2}] = 2$ and $[-\frac{5}{2}] = -3$. Then the association of $[x]$ to x is a real function on **R**.

What is the range of the greatest integer function?

(12) *Inverse Trigonometric Functions* : In the previous class you studied inverse trigonometric functions such as $\sin^{-1}x$, $\cos^{-1}x$, etc. These inverse functions, as you already know, are real functions The domain and range (confining only to Principal values) of the inverse trigonometric functions are as stated below :

Function	Domain	Range	Definition of the Function
$\sin^{-1}x$	$[-1, 1]$	$[\frac{-\pi}{2}, \frac{\pi}{2}]$	$y = \sin^{-1}x \Rightarrow x = \sin y$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$	$y = \cos^{-1}x \Rightarrow x = \cos y$
$\tan^{-1}x$	$(-\infty, \infty)$	$[\frac{-\pi}{2}, \frac{\pi}{2}]$	$y = \tan^{-1}x \Rightarrow x = \tan y$
$\cot^{-1}x$	$(-\infty, \infty)$	$(0, \pi)$	$y = \cot^{-1}x \Rightarrow x = \cot y$

Note : Read that $[-1, 1]$ is the *closed interval* whose end points are -1 and 1. In general, the closed interval $[a, b]$ is the set $\{x \in \mathbf{R}: a \leq x \leq b\}$. The open interval with end points a and b is denoted by (a, b). In other words, $(a, b) = \{x \in \mathbf{R} : a < x < b\}$. We have also intervals which are closed at only one end point.

For example, $(a, b] = \{x \in \mathbf{R} : a < x \leq b\}$
and $[a, b) = \{x \in \mathbf{R} : a \leq x < b\}$.

Remark

It is not possible to list all real functions. Some important ones among them have been mentioned in the above twelve examples. In section 2.3, methods of combining known functions to define new functions, will be described. By these methods, we arrive at a fairly large class of functions. Some more functions will be introduced as and when required.

EXERCISE 2.1

1. What is the domain of the function $\dfrac{x}{x^2-3x+2}$?
2. What is the domain of the function cosec x ?
3. How do we define $\tan^{-1} x$?
4. What are all the real numbers x such that $[x] = 2$?
5. What are all the values taken by the function $|x|$?
6. What is the range of the constant function 1 ?
7. What is the domain of $\sin^{-1} 2x$?
8. What is the domain of $\cos^{-1}(3x-1)$?
9. What is the domain of $\dfrac{\sin^{-1}x}{x}$?
10. What is the domain of $\tan^{-1}(2x+1)$?

2.2 Graphs of Real Functions

You have already seen in the earlier class how to draw graphs of real functions defined on an interval Now we quickly recall the graphs of some important functions.

1 The graph of the *constant function* 1 is drawn in Fig. 2.1 :

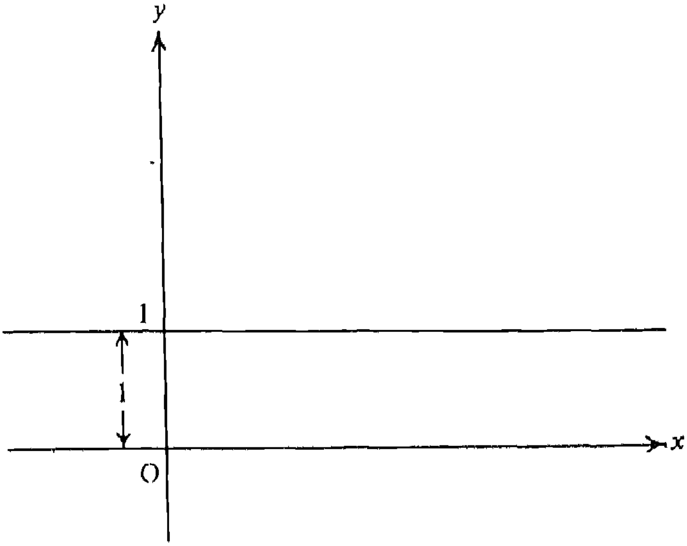


Fig. 2.1

We observe how the graph reflects some properties of the function.

Properties of the graph	*Properties of the function*
It is a line parallel to x-axis	It is a constant function
It is above the x-axis	The value of the function is positive
It is at a distance 1 from the x-axis	The value of the function has magnitude 1

For the constant function 0, the graph coincides with the x-axis. If the value of the constant function is negative, the graph will be a line below x-axis parallel to it.

2 The graph of the *identity function* x is a straight line through the origin, with slope 1 (See Fig. 2.2). We observe the following :

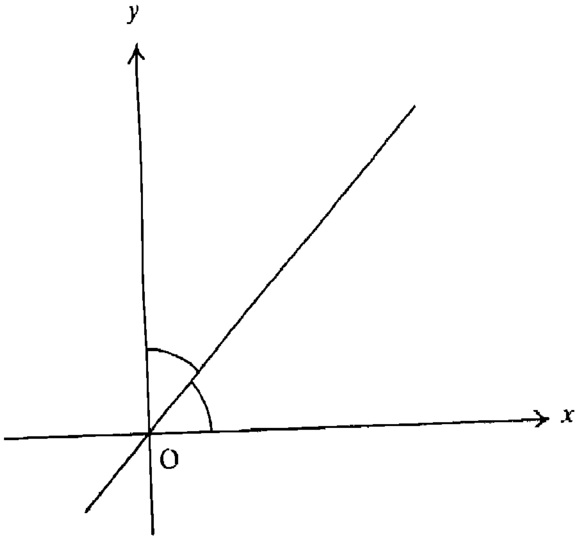


Fig. 2.2

Properties of the graph	*Properties of the function*
It is a straight line	It is of the form $mx + c$
It passes through origin	$c = 0$; that is, the value of the function at 0 is 0.
Its slope is 1	$m = 1$

3. The graph of the function x^2 is a parabola (Fig 2.3) whose vertex is at the origin and whose axis coincides with the y-axis.

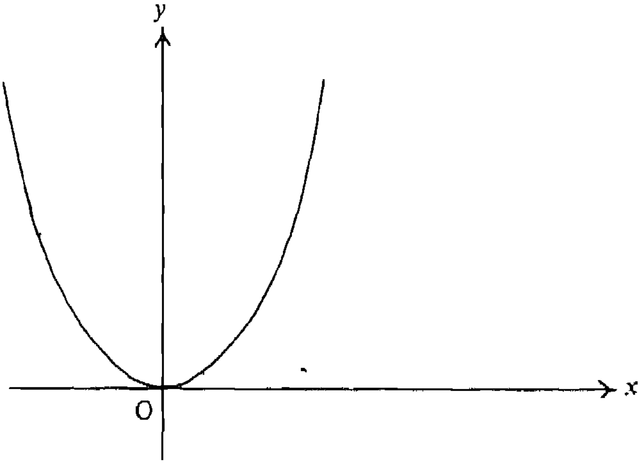


Fig. 2.3

Properties of the graph	*Properties of the function*
It is a parabola	It is a quadratic polynomial
It is above the x-axis	Only non-negative values are taken
It passes through origin	The value at 0 is 0
It is symmetric with respect to y-axis	It is an even function, that is, $f(-x) = f(x)$ for all x

The graph of the polynomial function $1 - x^2$ is a parabola shown below in Fig. 2.4.

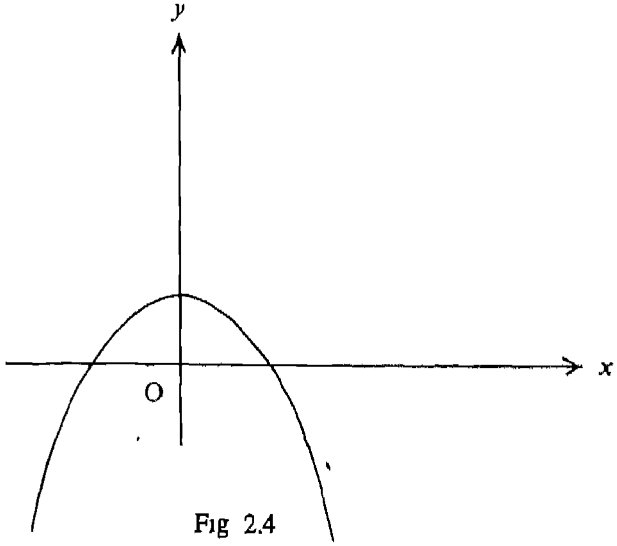


Fig 2.4

4. The graph of the *exponential function* e^x is shown below a in Fig. 2.5.

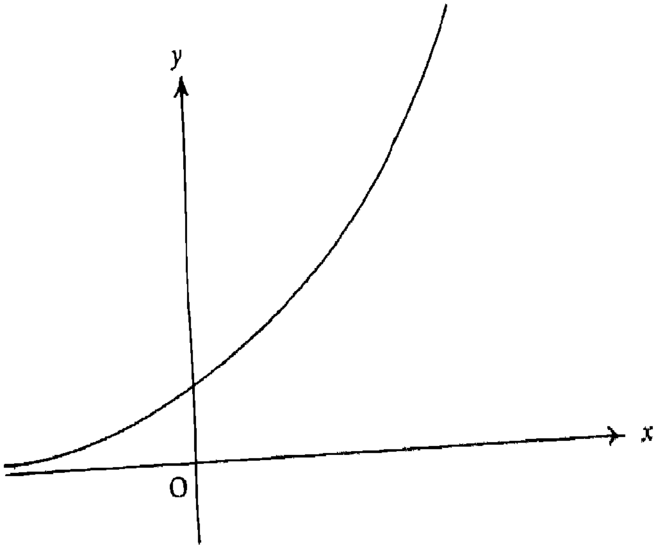


Fig. 2.5

We observe the following :

Properties of the graph	*Properties of the function*
As we see from left to right, the graph goes on rising above	It is an increasing function
It is fully above the x–axis.	All values taken are positive
In the second quadrant the graph comes nearer and nearer the x–axis but never touches it. (We say that it is asymptotic to the x–axis).	In the sense to be explained in § 3.7, $\lim_{x \to -\infty} e^x = 0$

5. The graph of the *logarithmic function* $\log x$ is given in Fig. 2.6 .

We observe the following :

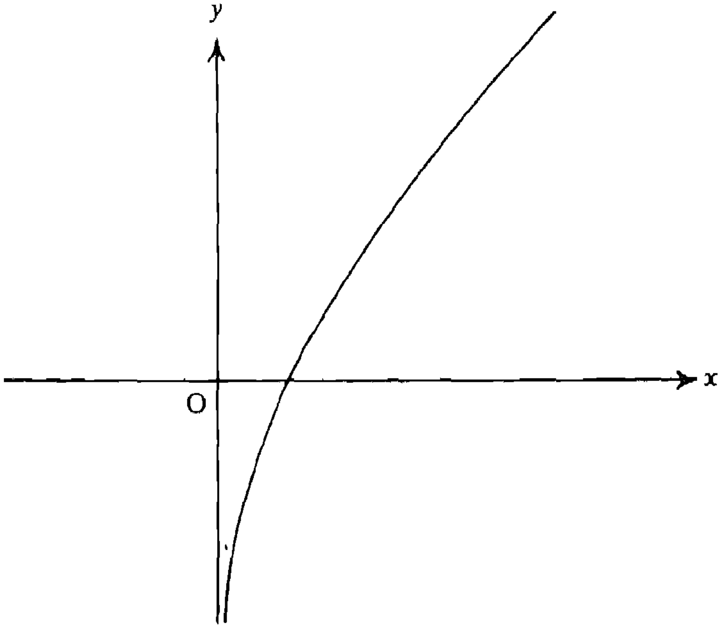


Fig. 2.6

Properties of the graph	*Properties of the function*
It is completely on the right side of the y–axis	The domain is the set of positive real numbers.
As we see from left to right the graph rises above	The function is increasing.
The left most part of the graph is coming nearer and nearer to the y–axis, but never meets it. (It is asymptotic to the y–axis)	$\lim\limits_{x \to 0} \log x = -\infty$

6. The graph of the *sine function, sin* x is drawn in Fig. 2.7 below .

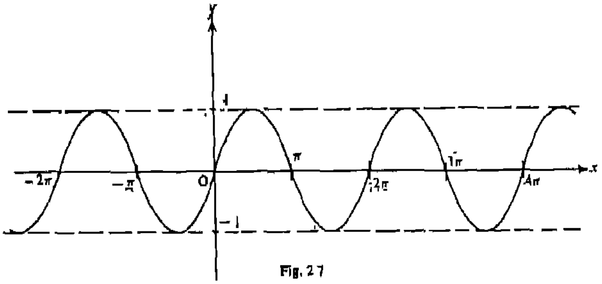


Fig. 2.7

We observe the following :

Properties of the graph	Properties of the function
The entire graph lies within a strip of distance 1 on each side of the x-axis	$-1 \le \sin x \le 1$ for all x
The same pattern repeats in intervals of equal length	The function is periodic with period 2π $\sin(x + 2\pi) = \sin x$ for all x
It passes through the origin	$\sin 0 = 0$

7. The graph of the *cosine function* is given in Fig 2.8 .

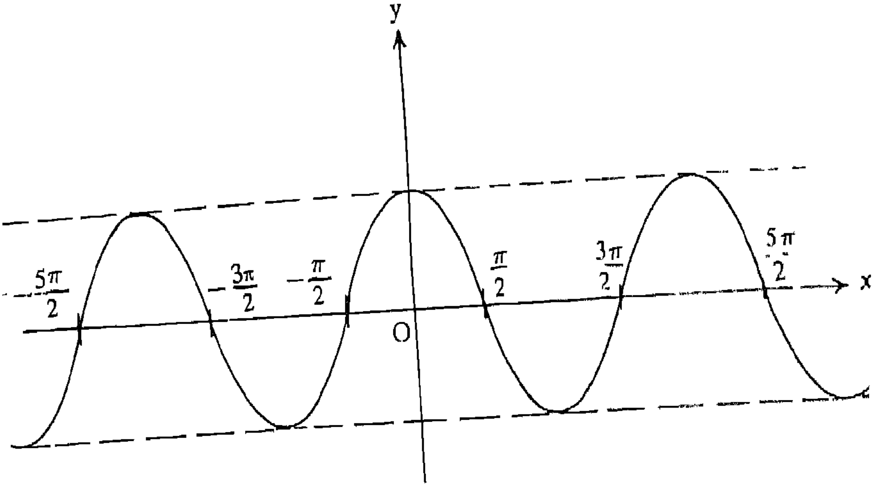


Fig. 2.8

We observe the following :

Properties of the graph	Properties of the function
It is symmetric with respect to y-axis	It is an even function because $\cos(-x) = \cos x$ for all x.
It looks like the graph of the sine function, the only difference being that it is shifted in the direction of the x-axis	It is related to the sine function by the rule $\sin(x + \frac{\pi}{2}) = \cos x$ for all x

8. The graph of the function tan x is drawn in Fig. 2.9.

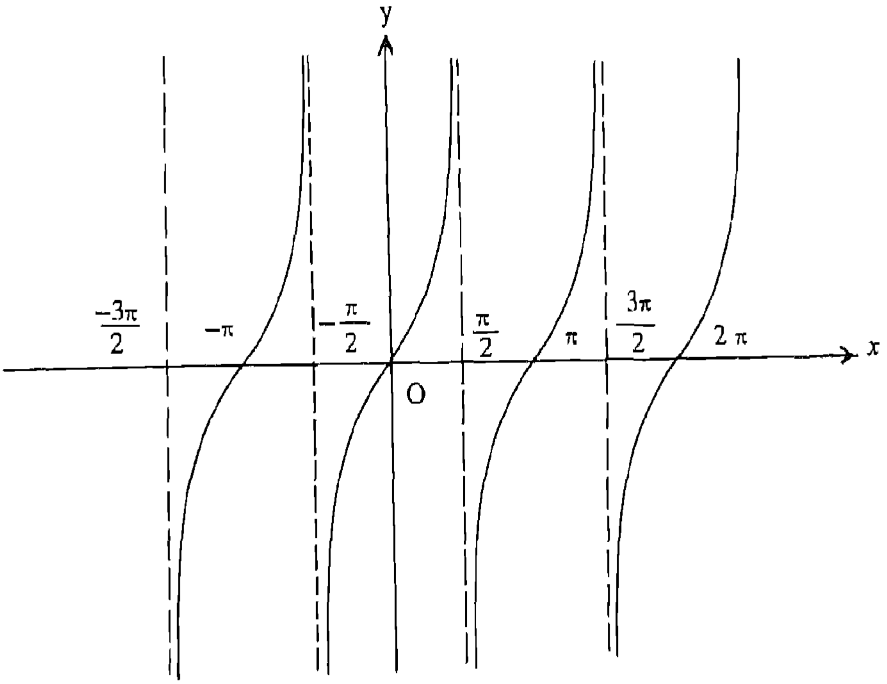


Fig 2.9

We observe the following :

Properties of the graph	*Properties of the function*
It is made up of several pieces	All odd multiples of $\frac{\pi}{2}$ are outside the domain
Same pattern repeats again and again	It is *periodic* with period π $\tan(x + \pi) = \tan x$ for all x
Each piece is *asymptotic* to a vertical line, one on the left and another on the right	$\lim_{x \to \frac{\pi}{2}-} \tan x = \infty$ and $\lim_{x \to \frac{\pi}{2}+} \tan x = -\infty$

9 The graph of the *modulus function*, $|x|$, consists of two rays starting from the origin as shown in Fig. 2.10.

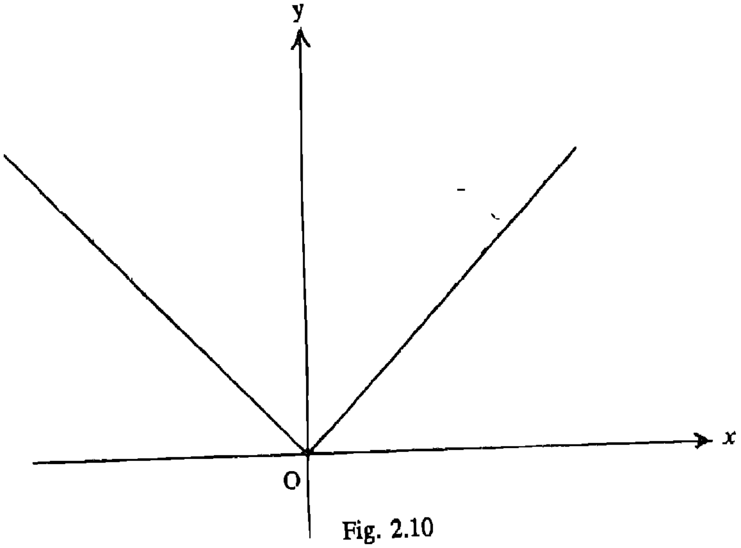


Fig. 2.10

We observe the following :

Properties of the graph	*Properties of the function*
It is symmetric with respect to y-axis	It is an even function because $\|-x\|=\|x\|$ for all x
It is fully above the x-axis	All values taken are non-negative
In the first quadrant, it coincides with the graph of the identity function	$\|x\| = x$ for all $x \geq 0$

10. The graph of the *greatest integer function*, $[x]$, is shown in Fig. 2.11 below.

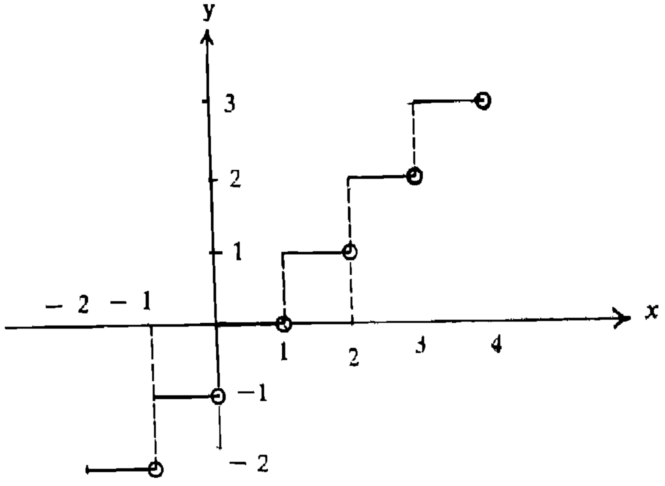


Fig. 2.11

We observe the following :

Properties of the graph	*Properties of the function*
It consists of many broken pieces	It is not continuous
Each piece coincides with the graph of a constant function	On each interval $[n, n + 1]$, the function takes the constant value n
The graph lies within the first and third quadrants	x and $[x]$ have same sign for all x

Remark

We have thus seen how the graph of a function reflects some of the important properties of the function. We have deliberately mentioned some properties involving limits that are to be studied in later sections. You are advised to skip them now, but read them after studying the relevant sections.

The theorems of differential calculus will be later seen to be helpful in sketching the graphs of functions.

EXERCISE 2.2

1. Draw the graph of the function $1-x$.
2. Draw the graph of the function $\frac{1}{x}$.
3. If the graph of a function does not meet the first and third quadrants, what is the corresponding property of that function?
4. If a function f has the property that $f(1 + x) = f(1 - x)$ for all x, how is this property reflected in its graph?
5. If f and g are two functions such that $f(x) \leq g(x)$ for all x, how is this property reflected in the graphs of these two functions?

2.3 Operations on Real Functions

In this section, we see how

two real functions can be added,
two real functions can be multiplied,
a real function can be multiplied by a number,
two real functions can be composed
and some real functions have inverses.

Sum : Let $f: X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be two real functions on a set X. Then we define their sum $f + g$ as that function from X to $\mathbf{R}$ which takes x in X to the number $f(x) + g(x)$. In other words,

$$(f+g)(x) = f(x) + g(x) \text{ for all } x \text{ in } X.$$

Remark

This is not a formula requiring a proof. This is our definition of $f + g$.

Roughly speaking, this means,

- value of the sum at x = sum of the values at x

For example, the sum of the identity function and the modulus function, is $x + |x|$, which can be alternately described as

$$f(x) = \begin{cases} 2x \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}$$

Product · Similarly the product of two functions, is also defined pointwise by the rule

$$(fg)(x) = f(x).\, g(x) \text{ for all } x \text{ in } X.$$

For example, the product of the identity function with itself is the function x^2

Remark

$f + g$ and fg are defined only when f and g are real functions having the same domain. In case, f and g have different domains, one can define their sum and product for those points x that are in the domains of both f and g.

Multiplication by a number :

Let $f \cdot X \to \mathbf{R}$ be a function and let c be a real number. Then we define the function $cf : X \to \mathbf{R}$ by the rule

$$(cf)(x) = cf(x) \text{ for all } x \text{ in } X.$$

For example, the greatest integer function $[x]$ when multiplied by 2 gives the function $2[x]$ whose value at any point is 2 times an integer.

Remark

Note that these operations are defined here only for real functions. For general functions from one set to another, these do not make sense.

Composition .

Suppose f and g are two functions such that the domain of f includes all elements of the form $g(x)$, where x is in the domain X of g. Then it makes sense to talk of $f(g(x)$ for each x in X. This $f(g(x)$ is a unique element associated to X in this way. Thus it defines a function with domain X

We denote this function by fog and call it the composite of f and g. In other words, fog is defined by the rule

$$(fog)(x) = f(g(x)) \text{ for all } x \text{ in } X.$$

For example, if f $\mathbf{R} \to \mathbf{R}$ is the function x^2 and $g \cdot \mathbf{R} \to \mathbf{R}$ is the function $\sin x$, then fog

is the function sin x^2, whereas gof is the function $\sin^2 x$.

Remarks

Note that fog is not always defined. For example, if f is the reciprocal function $\frac{1}{x}$ and y is the constant function zero, then fog is not defined. This is because the domain of f does not include all the values taken by g.

Even when both fog and gof are defined, they may not be equal.

Inverse Let $f : S \to T$ be a function such that f takes distinct values at distinct points of f. Then f is called an one-to-one function. Let the range of f be denoted by $f(S)$. Then for each y in $f(S)$, there is one and only one x in S such that $f(x) = y$ Thus we get a function from $f(S)$ to S. We denote it by f^{-1} and call it the inverse of f.

Note that the domain of f^{-1} is the range of f and that $f^{-1}(y)$ is that element x such that $f(x) = y$.

Note also that for each x in S, $f^{-1}(f(x)) = x$ and that for each y in S, $f(f^{-1}(y)) = y$.

It follows that the inverse of f^{-1} is f. In notation, $(f^{-1})^{-1} = f$.

As you can easily see the identity function is its own inverse. The inverse of the function $x+1$ is the function $x-1$. The function x^2 is not one-to-one, and so does not have an inverse.

Sometimes, we restrict the domain of a function. For example, sin x has the domain R. But, we can consider it as a function from $[\frac{-\pi}{2}, \frac{\pi}{2}]$ to R. when done so, it is an one-to-one function. So, it has an inverse We call this as the function $\sin^{-1}x$ It is a function from $[-1, 1]$ to $[\frac{-\pi}{2}, \frac{\pi}{2}]$.

Similarly, when x^2 is restricted to the set of positive real numbers, it has an inverse, namely, $\sqrt{x}$.

EXERCISE 2.3

1 Let $f : \mathbf{R} \to \mathbf{R}$ be the identity function. What is fof ?. What is ff ?
2. Let f be the sine function and let g be the function $2x$; what is fof ?. What is gog ?. Are these two the same ?
3. Let f be the exponential function and let g be the logarithmic function.

   What is $(f + g)(1)$?
   What is $(fg)(1)$?
   What is $(fog)(1)$?
   What is $(2f)(1)$?

4 Let f be the sine function, let g be the function $2x$ and let h be the cosine function. Prove that the function fog is the same as the function $go(fh)$.

5. Let f be any real function and let g be the function $2x$.
Then prove that gof is same as $f+f$.

6. Let f be the greatest integer function and g be the modulus function.
Prove that $(gof)(-\frac{5}{3}) - (fog)(-\frac{5}{3}) = 1$

7. If f and g are as in the above problem, what is $(gof)(\frac{5}{3}) - (fog)(\frac{5}{3})$?

8. If f and g are as in the above problem, what is $(f+2g)(-1)$?

2.4 Limit of a Function

Variable

Consider the example of the function $\sin x$.

It is a function from **R** to **R**.

When $x = 0$, its value is 0.

When $x = \frac{\pi}{2}$, its value is 1, and so on

We say that x is a variable here. For each value of the variable x, there is a value of the function.

Notation

Let x be the variable and let a be a fixed number. Then we use the notation $x \to a$ to mean that

x tends to a

or x approaches a.

What we mean is that the variable x is taking values very close to the value a (but not a itself).

Now, let f be a real function. When x takes values very close to a (and not exactly a), then the value $f(x)$ may or may not be very close to some number. In notation, as $x \to a$, it may (or may not) be true that $f(x) \to l$ for some number l. If it so happens that as $x \to a$, $f(x) \to l$, then we say that l is the limit of f as $x \to a$. We denote this by

$$\lim_{x \to a} f(x) = l.$$

We also say that the limit of the functions at a is l.

When there is no such l, we say that $\lim_{x \to a} f(x)$ does not exist or the limit at a does not exist.

Since the concept of limit is an abstract one, we present many concrete examples to illustrate it. The precise rigorous definition is, however, deferred to higher classes.

We start with the simplest kind of example.

Example 2.1

$\lim_{h \to 0} (2 + 3h - 4h^2) = 2$. This is because as $h \to 0$, h is very close to zero, and, therefore, negligibly small. So, we omit the terms $3h$ and $-4h^2$, and write the limit as 2.

Remark

More generally, if $p(h) = a_0 + a_1 h + a_2 h^2 + \ldots + a_n h^n$ is any polynomial, then

$$\lim_{h \to 0} p(h) = a_0.$$

We use this in the other examples below :

Example 2 2

Find $\lim_{x \to 1} x^2$.

Solution

Let $x = 1 + h$.

Then $x \to 1$ when and only when $h \to 0$.

$$\therefore \lim_{x \to 1} x^2 = \lim_{h \to 0} (1 + h)^2$$

$$= \lim_{h \to 0} (1 + 2h + h^2)$$

$$= 1.$$

Example 2.3 Find $\lim_{x \to 0} \frac{x^3-1}{x+2}$

Solution

We find the limits separately for the numerator and the denominator.

$$\lim_{x \to 0} (x^3 - 1) = -1$$

$$\lim_{x \to 0} (x + 2) = 2.$$

$$\text{Therefore, } \lim_{x \to 0} \frac{x^3 - 1}{x + 2} = -\frac{1}{2}.$$

In the next example, we see that it is not always so simple.

Example 2.4

Find $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$.

Here, if we separately take the limit for the numerator and the denominator, we see that both of them are zeroes Since $\frac{0}{0}$ is an indeterminate form, this method does not give the limit we want. We resort to some other method.

Put $x = 1 + h$.

$$\text{Then } \frac{x^3 - 1}{x - 1} = \frac{(1 + h)^3 - 1}{1 + h - 1} = \frac{1 + 3h + 3h^2 + h^3 - 1}{h}$$

$$= \frac{3h + 3h^2 + h^3}{h} = 3 + 3h + h^2$$

$$\therefore \lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{h \to 0} (3 + 3h + h^2) = 3$$

Another method : Using the formula

$$x^3 - 1 = (x - 1)(x^2 + x + 1), \text{ we have}$$

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} (x^2 + x + 1)$$

$$= 1 + 1 + 1 = 3.$$

Remark

In all the above examples, you may have noticed that the limit as $x \to a$ has coincided with the value at a.

In example 2.1, $\lim_{h \to 0} (2 + 3h - 4h^2) = 2$.

The value at $h = 0$ is also 2, as seen by substituting $h = 0$.

In example 2.2, $\lim_{x \to 1} x^2 = 1$.

The value at $x = 1$ is $1^2 = 1$.

In example 2.3, merely by sustituting $x = 0$ in the expression $\frac{x^3 - 1}{x + 2}$, one gets

$\frac{0^3 - 1}{0 + 2} = -\frac{1}{2}$. This is the same as our answer for the limit.

It was only in Example 2.4 that mere substitution posed some problem, yielding $\frac{0}{0}$. But by cancelling out the factor $x - 1$ that is common to both the numerator and the denominator, we rewrite the function as $x^2 + x + 1$ and then observe that the limit at 1 is the same as the value at 1.

This raises a pertinent question :

Is the limit same as the value ?

Is $\lim_{x \to a} f(x)$ same as $f(a)$?

It is not always so.

In the previous examples, it was so, because the functions were 'nice' functions

The distinction between limit and value

(a) The limit of a function at a point does not depend on the value at that point It depends only on the values taken at nearby points.
For example, for the function

$$f(x) = 1 \text{ if } x = 1$$
$$= 0 \text{ if } x \neq 1,$$

the value at 1 is 1, but the limit at 1 is 0.

(b) When a function $f(x)$ is given by a formula, usually the value is obtained by substituting $x = a$ in that formula, but the limit is not so.

Sometimes $f(a)$ may exist (that is, a is in the domain of f), but $\lim_{x \to a} f(x)$ may not exist (See Example 2.9 below)

Sometimes, $\lim_{x \to a} f(x)$ may exist, but a may not be in the domain of f (See Example 2.6 below).

Sometimes, both $\lim_{x \to a} f(x)$ and $f(a)$ may exist, but may not be equal. (See Example 2.5 below).

But such things happen only for functions that are not very interesting. In the next section, we shall define the notion of a continuous function and see that all polynomial functions and many other functions that are frequently encountered, are continuous. It turns out that for these functions the limit and the value are the same at any point

Example 2.5

Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x \text{ if } x \text{ is an integer} \\ 0 \text{ if } x \text{ is not an integer.} \end{cases}$$

Find $\lim_{x \to 1} f(x)$. Is it same as $f(1)$?

Solution

When x takes a value very close to 1 and different from 1, then x is not an integer. Therefore, $f(x)$ remains as zero for all such values of x.

Thus $\lim_{x \to 1} f(x) = 0$.

But $f(1) = 1$, since 1 is an integer.

Remark

Note that in the above example, the function f is not defined by a 'nice' formula, but defined in two pieces. Nevertheless, it is a real function and falls within the purview of our study in this section. You may recall that $|x|$ was also defined in two pieces like this.

Example 2.6

Find $\lim_{x \to 0} x \sin \frac{1}{x}$.

Solution

Note that, as it is, the function is not defined at $x = 0$. Nonetheless, the limit exists as $x \to 0$. This is because, $\sin \frac{1}{x}$ always lies between -1 and 1. Therefore, $x \sin \frac{1}{x}$ lies always between $-x$ and x. So $x \sin \frac{1}{x}$ must be at least as close to 0 as x is. Therefore, as $x \to 0$, $x \sin \frac{1}{x}$ also tends to 0. In other words, $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

The following fundamental facts about limits, will be freely used :

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$\lim_{x \to a} (c.f(x)) = c. \lim_{x \to a} f(x)$$

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

$$\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$$

$$\lim_{x \to a} (f(g(x))) = \lim_{x \to b} f(x) \text{ where } b = \lim_{x \to a} g(x)$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \text{ provided } \lim_{x \to a} g(x) \neq 0.$$

If $f(x) \leq g(x)$ for all x, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

Note : If $f(x) < g(x)$ for all x, we cannot conclude that $\lim_{x \to a} f(x) < \lim_{x \to a} g(x)$; we can conclude only that

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

The first of these states that the limit of the sum of two functions is the same as the sum of the limits. It must, however, be remarked that these equalities hold only when the concerned limits exist. For instance, it may happen that left side limit exists, but the right side limits do not exist (See example 2.8 below).

Similar remarks hold for each of these facts about limits.

Example 2.7

Find $$\lim_{x \to 2} \left[\frac{x^8-16}{x^4-4} + \frac{x^2-9}{x-3} \right].$$

Solution

$$\frac{x^8-16}{x^4-4} = \frac{(x^4-4)(x^4+4)}{x^4-4}$$
$$= x^4+4$$

$$\therefore \lim_{x \to 2} \frac{x^8-16}{x^4-4} = \lim_{x \to 2} (x^4+4) = 2^4+4 = 20.$$

Similarly,

$$\therefore \lim_{x \to 2} \frac{x^2-9}{x-3} = 5$$

$$\therefore \lim_{x \to 2} \left[\frac{x^8-16}{x^4-4} + \frac{x^2-9}{x-3} \right] = 20 + 5 = 25.$$

Example 2.8

Find $$\lim_{x \to 1} \left[\frac{1}{x-1} + \frac{1}{1-x} \right].$$

Solution

The function $\frac{1}{x-1}$ does not have any real number as limit at 1. So is the function $\frac{1}{1-x}$. But the sum $\frac{1}{x-1} + \frac{1}{1-x}$ is same as the zero function. Therefore, its limit at 1 is zero.

Example 2.9

Find $\lim_{x \to 3} [x]$.

Solution

The numbers 3.01, 3.002, 2.99, 2.998 are very close to 3. If we take the values at these points, they are 3,3,2,2. Not all of them are very close to 2. Not all of them are close to 3. Not all of them are close to any real number α. This limit does not exist. (See section 2.8 for more details).

Summary of Methods to find limits at a

For polynomial functions, rational functions, trigonometric functions, etc., we proceed as follows :

(1) Take the value at a. If it is not indeterminate, it is the limit at a. This will be justified in a latter chapter.

(2) If it is indeterminate, cancel out the common factor in the numerator and the denominator and then take the value at a.

(Example : Problems 5 and 9 in Exercise 2.4).

(3) Or expand the products, simplify, and take the value at a.
(Example : Problem 8 in Exercise 2.4)

(4) Sometimes conjugate surds can be used to simplify, and then the value at a may be taken.

(Example : Problem 10 in Exercise 2.4).

(5) Some standard formulas for finding limits can be applied.

(See Section 2 5, 2.6 and 2.9)

The following result on limits will be very useful later.

Theorem 2 1

Let f, g, h be three real functions on the same domain. Let $f(x) \leq g(x) \leq h(x)$ for all x in the domain. If $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} h(x)$ exist and are equal, then $\lim\limits_{x \to a} g(x)$ also exists and is equal to the limit of $f(x)$ (or $h(x)$).

Explanation

If a function is *between* two other functions known to have the same limits at a, then this function also has the same limit at a.

Remark

Though the proof is not difficult, we omit the proof. However, the result is important and will be used in section 2.6 and 2.9 to evaluate some important limits.

EXERCISE 2.4

Evaluate the following limits :

1. $\lim_{x \to 1} (x-1)^2 + 5$

2. $\lim_{x \to 0} (x-1)^2 + 5$

3. $\lim_{x \to 1} \dfrac{x-1}{x+1}$

4 $\lim_{x \to 1} \dfrac{x^2-1}{x-1}$

5. $\lim_{x \to -1} \dfrac{x^2-1}{x+1}$

6. $\lim_{x \to 3} (x^2-9) \left[\dfrac{1}{x+3} + \dfrac{1}{x-3} \right]$

7. $\lim_{x \to 0} \dfrac{ax+b}{cx+d}$

8. $\lim_{x \to 0} \dfrac{(1+x)^6-1}{(1+x)^2-1}$

9. $\lim_{x \to 0} \dfrac{x^4-3x^3+2}{x^3-5x^2+3x+1}$

10. $\lim_{x \to 2} \dfrac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$

2.5 A Special Limit

In this section we prove the formula

$$\lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}$$

and apply this to find limits of similar nature.

Theorem 2 2

Let n be any positive integer. Then

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1} .$$

We give two proofs of this theorem

First Proof : Put $x = a + h$. Then

$$\frac{x^n - a^n}{x - a} = \frac{(a + h)^n - a^n}{a + h - a} = \frac{1}{h} \left[(a + h)^n - a^n \right]$$

$$= \frac{1}{h} \left[a^n + \binom{n}{1} a^{n-1}h + \ldots + h^n - a^n \right]$$

(by Binomial Theorem)

$$= \frac{1}{h} \left[\binom{n}{1} a^{n-1} h + \binom{n}{2} a^{n-2} h^2 + \ldots + h^n \right]$$

$$= \binom{n}{1} a^{n-1} + \binom{n}{2} a^{n-2} h + \ldots + h^{n-1}$$

$$\therefore \quad \lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{h \to 0} \frac{(a+h)^n - a^n}{(a + h) - a}$$

$$= \lim_{h \to 0} \left[\binom{n}{1} a^{n-1} + \binom{n}{2} a^{n-2} h + \ldots + h^{n-1} \right]$$

$$= \binom{n}{1} a^{n-1}$$

$$= n\, a^{n-1}.$$

Second Proof : We have the formula

$$x^n - a^n = (x - a) (x^{n-1} + a x^{n-2} + a^2x^{n-3} + \ldots + a^{n-2} x + a^{n-1}).$$

(This can be proved easily. Just multiply the right side and see that many terms cancel each other, and what remains is $x^n - a^n$).

Therefore, $$\frac{x^n - a^n}{x - a} = x^{n-1} + a x^{n-2} + a^2x^{n-3} + \ldots + a^{n-2}x + a^{n-1}$$

$$\therefore \quad \lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \left[x^{n-1} + a x^{n-2} + a^2x^{n-3} + \ldots + a^{n-2} x + a^{n-1} \right]$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \ldots + a^{n-1} \ (n \text{ terms})$$

$$= n\,a^{n-1}.$$

Remark

In the above theorem, we assumed that n is a positive integer. The formula remains true, when n is a positive rational number, (for instance $\frac{1}{2}$) and if a is positive.

Here, why do we assume that a is positive ? It is because, if a were negative, $a^{\frac{p}{q}}$ is not defined except when $\frac{p}{q}$ is an integer. For instance, $(-1)^{\frac{1}{2}}$ is not defined. (Note : We are considering only real numbers in this course, not complex numbers).

It is now natural to ask whether the formula remains true even when $\frac{p}{q}$ is negative. The answer is affirmative.

Thus we have

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

holds for all rational numbers n, provided either a is positive, or n is an integer. We omit the proof of this. But we use this result in later sections also.

Example 2.10

Evaluate $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x^2 - 4}$.

Solution

First Method : We use the formula $\lim\limits_{x \to a} \left(\dfrac{x^n - a^n}{x - a} \right) = na^{n-1}$.

$$\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \to 2} \left(\frac{x^3 - 8}{x - 2} + \frac{x^2 - 4}{x - 2} \right)$$

$$= \lim_{x \to 2} \frac{x^3 - 2^3}{x - 2} + \lim_{x \to 2} \frac{x^2 - 2^2}{x - 2}$$

$$= 3(2)^2 + 2(2)$$

$$= 12 + 4$$

$$= 3.$$

Second Method : The numerator and the denominator have a common factor, $x - 2$. Cancelling this,

$$\frac{x^3-8}{x^2-4}=\frac{x^2+2x+4}{x+2}$$

$$\therefore \lim_{x\to 2}\frac{x^3-8}{x^2-4}=\frac{2^2+2(2)+4}{2+2}=\frac{12}{4}=3.$$

Example 2.11

Find $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}$.

Solution

Put $y=1+x$ Then

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{y\to 1}\frac{\sqrt{y}-1}{y-1}\quad \text{(because as } x\to 0,\ y\to 1)$$

$$=\lim_{y\to 1}\frac{y^{\frac{1}{2}}-1^{\frac{1}{2}}}{y-1}$$

$$=\frac{1}{2}\,1^{-\frac{1}{2}}=\frac{1}{2}\,.$$

EXERCISE 2.5

1. Show that $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{x}=n.$

2. Find $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$.

3. Find $\lim\limits_{x\to 1}\dfrac{1-x^{-\frac{1}{3}}}{1-x^{-\frac{2}{3}}}$.

4 Evaluate $\lim\limits_{x\to -1}\dfrac{x^3+1}{x+1}$.

5. If $\lim\limits_{x\to a}\dfrac{x^9+a^9}{x+a}=9$, find all possible values of a.

6. If $\lim\limits_{x\to 1}\dfrac{x^4-1}{x-1}=\lim\limits_{x\to k}\dfrac{x^3-k^3}{x^2-k^2}$, find the value of k.

7. If $\lim\limits_{x\to 2}\dfrac{x^n-2^n}{x-2}=80$ and if n is a positive integer, find n.

8. Evaluate $\lim\limits_{x\to a}\dfrac{(x+2)^{\frac{5}{3}}-(a+2)^{\frac{5}{3}}}{x-a}$

2.6 Some Trigonometric Limits

In this section, we find the limits of some trigonometic functions at some points. Mainly, we prove $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$. Then we apply this to do some more exercises on limits.

Theorem 2 3

$\lim\limits_{\theta\to 0}\sin\theta=0$

and $\lim\limits_{\theta\to 0}\cos\theta=1$.

Proof · Draw a right-angled triangle ABC where $\angle A=\theta$ and $\angle B$ is a right angle (Fig. 2.12). Then $\sin\theta=\dfrac{BC}{AC}$ and $\cos\theta=\dfrac{AB}{AC}$.

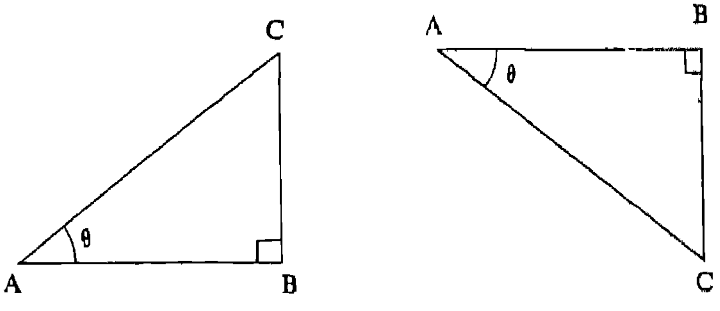


Fig. 2.12

Now, imagine that θ becomes smaller and smaller; keep the side AB fixed; vary C alone on the same line BC, bringing it nearer and nearer to B. We find that if θ is very small, then AC is very close to AB and further BC is very small. Therefore, as $\theta\to 0$, $\dfrac{BC}{AC}$ also approaches 0 and $\dfrac{AB}{AC}$ approaches 1. Therefore, as $\theta\to 0$ $\sin\theta\to$ and $\cos\theta\to 1$.

Remark

The above theorem says that the limit is the same as the value, for the sine and cosine functions, at zero. This will be used in the proof of the next theorem.

Theorem 2.4

$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, when θ is measured in radians.

Note

If we find the limits for the numerator and denominator separately, we end up with the indeterminate form So we need another method of finding this limit.

Proof

First we prove that $\frac{\sin \theta}{\theta}$ always lies between $\cos \theta$ and 1.

$$\text{Since, } \frac{\sin(-\theta)}{-\theta} = \frac{\sin \theta}{\theta}$$

$$\text{and } \cos(-\theta) = \cos \theta,$$

it is enough to prove this for positive values of θ.

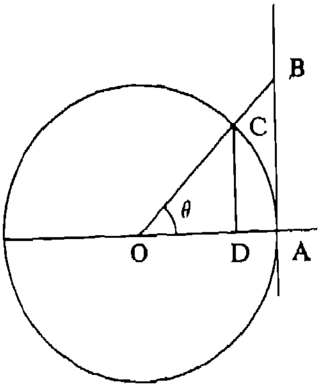


Fig. 2.13

In the figure 2.13, OAB is a triangle with $\angle O = \theta$ radians and $\angle A = 90°$. Take a point C on the hypotenuse OB such that $OC = OA$. Draw a line CD parallel to AB meeting OA at D. Draw the circle with centre O and radius OA and note that it passes through C, because $OA = OC$.

Now, area of $\Delta OAC <$ area of sector $OAC <$ area of ΔOAB

This gives

$$\frac{1}{2} OA.CD < \frac{1}{2} OA^2.\theta < \frac{1}{2} OA.AB$$

This implies

$$CD < OA.\theta < AB.$$

Noting that $CD = OC \sin \theta = OA \sin \theta$

and $AB = OA \tan\theta$, we get

$$OA \sin\theta < OA.\theta < OA \tan\theta$$

and, therefore, $\sin\theta < \theta < \tan\theta$.

Now dividing by the positive quantity $\sin\theta$, we get

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocals, we have,

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Having proved this inequality, we turn to our problem on limits. Taking limits as $\theta \to 0$, we get

$$\lim_{\theta\to 0} 1 \geq \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \geq \lim_{\theta\to 0} \cos\theta.$$

(See Section 3.4 where we have stated that if $f(x) < g(x)$ for all x, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)\ )$$

We already know that $\lim\limits_{\theta\to 0} 1 = 1$ and $\lim\limits_{\theta\to 0} \cos\theta = 1$ (by previous theorem).

Therefore, $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}$ lies between 1 and 1

$\therefore$ limit has to be equal to 1.

$$\therefore \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

Remark

It is important to note here that the real number θ is the measure of the angle in radians. Otherwise, the limit may not be 1.

Example 2 12

Find $\lim\limits_{\theta\to 0} \dfrac{\sin 2\theta}{\sin 3\theta}$

Solution

$$\lim_{\theta\to 0} \frac{\sin 2\theta}{\sin 3\theta} = \lim_{\theta\to 0} \frac{\sin 2\theta}{2\theta} \cdot \frac{2\theta}{3\theta} \cdot \frac{3\theta}{\sin 3\theta}$$

$$= \left[\lim_{2\theta\to 0} \frac{\sin 2\theta}{2\theta}\right] \left[\frac{2}{3}\right] \div \left[\lim_{3\theta\to 0} \frac{\sin 3\theta}{3\theta}\right]$$

$$= 1\left(\frac{2}{3}\right) \div 1$$

$$= \frac{2}{3}.$$

Example 2 13

Find $\lim\limits_{x\to\frac{\pi}{2}} (\sec x - \tan x)$.

Solution

Put $y = \frac{\pi}{2} - x$.

So, as $x \to \frac{\pi}{2}$, $y \to 0$

$$\therefore \lim_{x\to\frac{\pi}{2}} (\sec x - \tan x) = \lim_{y\to 0}\left[\sec\left(\frac{\pi}{2} - y\right) - \tan\left(\frac{\pi}{2} - y\right)\right]$$

$$= \lim_{y\to 0}\left[\frac{1-\cos y}{\sin y}\right]$$

$$= \lim_{y\to 0} \frac{2\sin^2\frac{y}{2}}{2\sin\frac{y}{2}\,.\,\cos\frac{y}{2}}$$

$$= \lim_{y\to 0} \frac{\sin\frac{y}{2}}{\cos\frac{y}{2}}$$

$$= 0 \div 1$$

$$= 0.$$

EXERCISE 2.6

Evaluate the limits in Problems 1 to 8 :

1. $\lim\limits_{\theta \to 0} \dfrac{1-\cos 4\theta}{1-\cos 5\theta}$

2. $\lim\limits_{\theta \to 0} \dfrac{1-\cos m\theta}{1-\cos n\theta}$ (Here m and n are fixed non-zero real numbers)

3. $\lim\limits_{\theta \to 0} \dfrac{\sin a\theta}{\sin b\theta}$ (Here a and b are fixed non-zero real number)

4. $\lim\limits_{\theta \to 0} \dfrac{\tan \theta}{\theta}$

5. $\lim\limits_{\theta \to 0} \theta \operatorname{cosec} \theta$

6. $\lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{\cot \theta}{\frac{\pi}{2} - \theta}$

7. $\lim\limits_{\theta \to 0} \dfrac{1-\cos\theta}{\theta}$

8. $\lim\limits_{\theta \to 0} \dfrac{1-\cos \theta}{\theta^2}$

9. If $\lim\limits_{\theta \to 0} k\theta \operatorname{cosec}\theta = \lim\limits_{\theta \to 0} \theta \operatorname{cosec} k\theta$, prove that k must be ± 1.

10. If θ is measured in degrees (not in radians), then what is $\lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta}$?

2.7 Limits at Infinity and Infinite Limits

In this section we explain the notations $x \to \infty$, $x \to -\infty$, and methods of finding the limits of a function of x as $x \to \infty$ or as $x \to -\infty$.

Notation

The symbol $x \to \infty$ will be used to mean that x takes very large values.

Example 2.14

Let $f(x) = \dfrac{1}{|x|}$.

Then as x becomes very small in magnitude, $f(x)$ becomes very large. We can say this using our notations as follows :

$$\text{As } x \to 0, \ \frac{1}{|x|} \to \infty .$$

We also write this as .

$$x \to 0 \ \frac{1}{|x|} = \infty \qquad ..(2.1)$$

Note once again that the symbol ∞ when it stands alone, has been given no meaning. It is not a number. Still (2.1) has a meaning. It means : when x is very close to 0, $\frac{1}{|x|}$ is very large.

Example 2.15

$\lim\limits_{x \to \infty} \frac{1}{x} = 0$. As x becomes very large, $\frac{1}{x}$ is very close to 0.

Example 2.16

As x becomes large, we know that x^2 also becomes large. We write this in the form :

as $x \to \infty$, $x^2 \to \infty$ or in the form : $\lim\limits_{x \to \infty} x^2 = \infty$.

Example 2.17

What is $\lim\limits_{x \to \infty} \sin x$?

Solution

For large values of x, what is the value that $\sin x$ approaches ?

It is not becoming very large, because $\sin x$ never exceeds 1.

When x takes the values, $\pi, 2\pi, 3\pi, \ldots, n\pi, \ldots$

even for large values of n, $\sin x$ becomes 0.

On the other hand, $\sin (2n\pi + \frac{\pi}{2}) = 1$ even for large values of n.

Therefore, there is no definite quantity to which $\sin x$ approaches, as x becomes large.

$$\therefore \lim_{x \to \infty} \sin x \text{ does not exist.}$$

Remark

One method to calculate limits at ∞ is to use the substitution $y = \frac{1}{x}$ and then take the limit at 0, as shown in the following example :

Example 2.18

Find $\lim\limits_{x \to \infty} \frac{x-1}{x+1}$.

Solution

$$\lim_{x \to \infty} \frac{x-1}{x+1} = \lim_{x \to \infty} \frac{1-\frac{1}{x}}{1+\frac{1}{x}}$$

$$= \lim_{y \to 0} \frac{1-y}{1+y}, \text{ where } y = \frac{1}{x}$$

$$= \frac{1-0}{1+0} = 1.$$

Notation

The symbol $x \to -\infty$ will be used to mean that x is a negative number having a large magnitude. $\lim_{x \to -\infty} f(x) = a$ means if x is the negative of a large quantity, then $f(x)$ is close to a.

Example 2 19

What is $\lim_{x \to -\infty} e^x$?

Solution

Put $y = -x$.

Then as $x \to -\infty$, $y \to \infty$.

$\therefore \lim_{x \to -\infty} e^x = \lim_{y \to \infty} e^{-y}$

but, for all positive y, we have $e^y \geq 1 + y$.

Thus, $e^{-y} = \frac{1}{e^y} \leq \frac{1}{1+y}$

Since, $\lim_{y \to \infty} \frac{1}{1+y} = 0$, we have $\lim_{y \to \infty} e^{-y} = 0$.

$\therefore \lim_{x \to -\infty} e^x = 0.$

EXERCISE 2.7

1. Find $\lim\limits_{x\to\infty} \dfrac{1}{x^2}$.

2. Evaluate $\lim\limits_{x\to\infty} \dfrac{1}{(1-x)^2}$.

3. Prove that $\lim\limits_{x\to\infty} \dfrac{x-1}{x} = 1$.

4 Prove that $\lim\limits_{x\to\infty} \dfrac{x^2+ax+b}{x^2+px+q}$ is a number independent of a, b, p and q.

(i.e. it is the same number, even when a, b, p, q are changed)

5 Prove that $\lim\limits_{x\to 1} \dfrac{1}{|1-x|} = \infty$.

6. What is $\lim\limits_{x\to\infty} e^{-x}$?

7. It is given that $f(x) = \dfrac{ax+b}{x+1}$, $\lim\limits_{x\to 0} f(x) = 2$

and $\lim\limits_{x\to\infty} f(x) = 1$. Prove that $f(-2) = 0$.

8. Find $\lim\limits_{x\to\infty} \dfrac{(3x-1)(4x-2)}{(x+8)(x-1)}$.

9. Prove that $\lim\limits_{x\to\infty} (\sqrt{x^2+x+1}-x) \neq \lim\limits_{x\to\infty} (\sqrt{x^2+1-x})$

2.8 One-Sided Limits

In this section, we explain the notations

$x \to a+$ and $x \to a-$

and discuss the relation between these one-sided limits, with the limits that we already know.

Notation

Let a be a real number. Recall that $x \to a$ means x takes a value very close to a.

Then the values taken by x can be on either side of a.

They may be greater than a or less than a, but are near a.

Now we introduce the notation $x \to a+$ to mean that x takes values near a and greater than a

The notation $x \to a-$ on the other hand means that x takes values near a and less than a.

Example 2.20

As $x \to 0+$, $\frac{1}{x} \to \infty$.

This means : As x takes very small positive values, $\frac{1}{x}$ takes very large values.

It is not true that as $x \to 0$, $\frac{1}{x} \to \infty$. Because, when x takes negative values close to 0, then $\frac{1}{x}$ is negative and, therefore, not large. In fact as $x \to 0-$, $\frac{1}{x} \to -\infty$.

Example 2.21

Prove that

$$\lim_{x \to 0+} \frac{x}{|x|} = 1$$

and

$$\lim_{x \to 0-} \frac{x}{|x|} = -1.$$

Solution

When x takes positive values, $\frac{x}{|x|}$ is 1.

When x takes negative values, $\frac{x}{|x|} = \frac{x}{-x} = -1$.

Therefore,

$$\lim_{x \to 0+} \frac{x}{|x|} = 1$$

and

$$\lim_{x \to 0-} \frac{x}{|x|} = -1.$$

Left limits and Right limits

$\lim\limits_{x \to a+} f(x)$ is called the right limit of f at a.

$\lim\limits_{x \to a-} f(x)$ is called the left limit of f at a.

Both these are called the limit of f at a.

$\lim\limits_{x \to a} f(x)$ is called the limit of f at a.

Theorem 2.5

For a real function f and a real number a, the following are equivalent :

(a) $\lim\limits_{x \to a} f(x) = \infty$

(b) $\lim\limits_{x \to a+} f(x) = \infty = \lim\limits_{x \to a-} f(x)$.

Proof :

$\lim\limits_{x \to a} f(x) = \infty$ means :

If x is very close to a (no matter whether greater than a or less than a), $f(x)$ is close to ∞.

This means two things :

If x is close to a and greater than a, then $f(x)$ is close to ∞ and if x is close to a and less than a, then $f(x)$ is close to ∞.

In other words, $\lim\limits_{x \to a+} f(x) = \infty = \lim\limits_{x \to a-} f(x)$.

Corollary

If $\lim\limits_{x \to a+} f(x) \neq \lim\limits_{x \to a-} f(x)$, then $\lim\limits_{x \to a} f(x)$ does not exist.

This corollary gives a useful way to prove the nonexistence of limits. If we prove that the right limit at a point is different from the left limit at that point, then the function does not have a limit at that point,

For example, we have already seen that the function $\frac{x}{|x|}$ has different left and right limits at 0. Therefore,

$\lim\limits_{x \to 0} \frac{x}{|x|}$ does not exist.

Remark

If a function is considered only on an internal the one sided limits at the end points of the interval, will be called (for convenience) as the limits at these end points.

Example 2.22

Prove that $\lim\limits_{x \to 1} [x]$ does not exist.

Solution

We first note that $\lim_{x \to 1+} [x] = 1$ This is because if x is close to 1 and greater than 1 (for instance 1.01, 1.002 etc.), then $[x] = 1$.

Next we note that $\lim_{x \to 1-} [x] = 0$. This is because if x is close to 1 and less than 1 (for instance 0.99, 0.998 etc.), then $[x] = 0$.

Thus the left limit at 1 and the right limit at 1 are unequal for the function $[x]$. Therefore, at $\lim_{x \to 1} [x]$ does not exist.

EXERCISE 2.8

1. Find $\lim_{x \to \frac{5}{2}} [x]$

2. Find $\lim_{x \to 3+} \frac{x}{[x]}$. Is it equal to $\lim_{x \to 3-} \frac{x}{[x]}$?

3. If f is a real function such that $f(x) = f(-x)$ for all x, then prove that $\lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x)$ whenever they exist. (Such functions are called even functions).

4. If f is an odd function and if $\lim_{x \to 0+} f(x)$ exists, prove that this limit must be zero. ('f is odd' means $f(-x) = -f(x)$ for all x)

2.9 Some Important Limits

Theorem 2.6

Prove that
$$\lim_{x \to 0+} f(-x) = \lim_{x \to 0-} f(x).$$

Proof : Let $y = -x$.

Then as $x \to 0-$, $y \to 0+$.

$$\therefore \lim_{x \to 0-} f(x) = \lim_{y \to 0+} f(-y)$$

$$= \lim_{x \to 0+} f(-x) \qquad \text{(by merely changing the variable throughout)}$$

Theorem 2.7

$$1 \le e^x \le 1 + \frac{x}{ex} \text{ for all } x \text{ such that } 0 \le x \le 1.$$

Proof : We know $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$

If x is positive, clearly, $e^x = 1 + \text{a positive quantity}$.

$\therefore$ $e^x > 1$ and if $x = 0$, then $e^x = 1$. Therefore, if $x \ge 0$, $e^x \ge 1$.

Also

$$1 + ex = 1 + x\left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots\right)$$

$$= 1 + x + \frac{x}{1!} + \frac{x}{2!} + \frac{x}{3!} + \dots$$

$$\ge 1 + x + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= x + \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots\right)$$

(Since $x \ge x^n$ whenever $0 \le x \le 1$)

$$= x + e^x$$

$$\ge e^x \text{ (Since } x \text{ is positive)}$$

Hence, $1 \le e^x \le 1 + ex$ for all x such that $0 \le x \le 1$.

Theorem 2.8

$$\lim_{x \to 0} e^x = 1.$$

Proof :

From Theorem 2.7, we have $1 \le e^x \le 1 + ex$ for all x such that $0 \le x \le 1$.

Taking limits as $x \to 0+$, we have

$$\lim_{x \to 0+} 1 \le \lim_{x \to 0+} e^x \le \lim_{x \to 0+} (1 + ex)$$

$\lim\limits_{x \to 0+} 1 = 1$ and $\lim\limits_{x \to 0+} (1 + ex) = 1.$

Therefore, $\lim\limits_{x \to 0+} e^x = 1.$

Again, we have from Theorem 2.6

$$\lim_{x \to 0+} f(-x) = \lim_{x \to 0-} f(x).$$

Taking $f(x) = e^x$, we have

$$\lim_{x \to 0-} e^x = \lim_{x \to 0+} e^{-x} \quad \therefore \ (f(-x) = e^{-x})$$

$$= \lim_{x \to 0+} \frac{1}{e^x}$$

$$= \frac{1}{\lim\limits_{x \to 0+} e^x}$$

$$= \frac{1}{1} \qquad (\lim_{x \to 0+} e^x = 1 \text{ proved above})$$

Thus, $\lim\limits_{x \to 0+} e^x = \lim\limits_{x \to 0-} e^x = 1.$

Therefore, $\lim\limits_{x \to 0} e^x = 1.$

Theorem 2.9

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

Proof :

As a first step, we shall prove the inequality

$$1 + x \leq e^x \leq 1 + x + x^2 (e - 2), \text{ if } 0 \leq x \leq 1. \qquad ...(2.2)$$

Now, $1 + x \leq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$ because $0 \leq x \leq 1$

$$= e^x$$

$$= 1 + x + x^2 \left[\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \ldots \right]$$

$$\le 1+x+x^2\left[\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots\right] \text{ (because } x\le 1)$$
$$=1+x+x^2(e-2).$$

Our second step is to deduce the inequality

$$1\le \frac{e^x-1}{x}\le 1+x(e-2), \text{ if } 0<x\le 1.$$

This follows from (2.1) by substracting 1 from each of the three quantities and dividing by the positive x.

We know $\lim\limits_{x\to 0} 1+x(e-2)=1.$

Therefore, it follows from the last Theorem of 2.4 that $\lim\limits_{x\to 0}\frac{e^x-1}{x}=1.$

Theorem 2.10

$$\lim_{x\to 0}\frac{\log(1+x)}{x}=1.$$

Proof :

Put $y=\log(1+x)$.
As $x\to 0$, $y\to 0$.

Now, $\lim\limits_{x\to 0}\frac{\log(1+x)}{x}=\lim\limits_{y\to 0}\frac{y}{e^y-1}=\frac{1}{\lim\limits_{y\to 0}\frac{e^y-1}{y}}=1$ (by previous theorem)

Remark

Here, we are assuming the result that $\lim\limits_{x\to 0}\log(1+x)=0$ without proof. When we substitute $x=0$, we get the value 0.

EXERCISE 2.9

1. Prove the inequality $0\le\log(1+x)\le x$ for all positive x. Deduce that $\lim\limits_{x\to 0+}\log(1+x)=0$.

2. Prove that $e^x\le 1+x$ for negative x. Deduce that $x\le\log(1+x)<0$ if x is negative. Deduce that $\lim\limits_{x\to 0-}\log(1+x)=0$.

3. Prove that $\lim\limits_{x\to 1} e^x=e$.

$$\left[\text{Hint}: \lim_{x\to 1} e^x = \lim_{y\to 0} e^{y+1}.\right]$$

4. Prove that $\lim_{x\to e} \log x = 1.$

$$\left[\text{Hint}: \text{Put } y = \frac{x}{e}\right]$$

2.10 Continuous Functions : In this section we define the concept of a continuous function and prove that some well-known functions are continuous.

Definition

Let f be a real function and let a be in the domain of f. We say that f is continuous at a, if the limit at a is same as the value at a, or in other words, $\lim_{x\to a} f(x)$ exists and equals $f(a)$

Many results proved in earlier sections can be restated, using the term 'continuous'. We list a few of them below :

$2+3x-4x^2$ is continuous at 0 (Example 2.1)

x^2 is continuous at 1 (Example 2.2)

(We shall later prove the stronger result that every polynomial is continuous at every point.)

$\frac{x^3-1}{x+2}$ is continuous at 0. (Example 2.3)

(We shall later prove that every rational function is continuous at every point of its domain).

$\sin x$ is continuous at 0 (Theorem 2.3)

$\cos x$ is continuous at 0 (Theorem 2.3)

e^x is continuous at 0 (Example 2.24)

$\log(1+x)$ is continuous at $x=0$ (Exercise 2.9[2])

e^x is continuous at 1 (Exercise 2.9[3])

$\log x$ is continuous at e (Exercise 2.9[4])

(We shall later prove that these functions $\sin x$, $\cos x$, e^x, $\log x$ and $\log(1+x)$ are continuous at every point of the domain).

$[x]$ is not continuous at 1 (Example 2.22)

$[x]$ is not continuous at 3 (Example 2.9)

The function $f(x) = \begin{cases} x \text{ if } x \text{ is an integer} \\ 0 \text{ if } x \text{ is not an integer} \end{cases}$

is not continuous at 1, 2, 3... (Example 2.5).

$\frac{x}{|x|}$ is not continuous at 0 (Remark after corollary of Theorem 2.5)

$\frac{x}{|x|}$ is not continuous at 3 (Exercise 2.8[2])

Definition

A real function is said to be continuous in an (open or closed) interval if it is continuous at every point of the interval.

Explanation

When a function f is considered on a closed interval $[a, b]$, then f is said to be continuous at the end point a if $\lim\limits_{x \to a+} f(x) = f(a)$.

Similarly, f is said to be continuous at the end point b if $\lim\limits_{x \to b-} f(x) = f(b)$.

When we say that a real function f is continuous (without mentioning where) we mean that it is continuous at every point of its domain.

Example 2.23

Every constant function is continuous. Consider the constant function $f(x) = \alpha$ for all x. We know that as x takes values close to a, $f(x)$ takes the value α. Therefore, $\lim\limits_{x \to a} f(x) = \alpha$

Also $f(a) = \alpha$.

$\therefore \lim\limits_{x \to a} f(x) = f(a)$. This means, f is continuous at a. This holds for all a in **R**. Therefore, f is continuous.

Example 2.24

The identity function is continuous.

Proof : To prove continuity of the identity function at a point a, we must prove

$$\lim_{x \to a} x = a$$

It is obvious then as $x \to a$, we have the function $x \to a$.

Thus the function x is continuous at a, for all a in **R**.

Thus it is continuous.

The following theorems will be useful to prove the continuity of functions.

Theorem 2.11

Let f and g be continuous at a. Then $f + g$ is continuous at a.

Proof : Since f is continuous at a,

$$\lim_{x \to a} f(x) = f(a).$$

Since g is continuous at a,

$$\lim_{x \to a} g(x) = g(a).$$

Adding these two,

$$\lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a).$$

But we have already seen in the previous chapter that

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

It follows that

$$\lim (f(x) + g(x)) = f(a) + g(a).$$

In other words, the function $f + g$ is continuous at a.

Corollary

If f and g are continuous, then $f + g$ is also continuous.

Restatement

Sum of two continuous functions is continuous. Similary, one can prove :

Theorem 2.12

Let f and g be two functions, continuous at a. Let c be a real number.

Then i) $f - g$ is continuous at a

ii) cf is continuous at a

iii) fg is continuous at a

iv) $\frac{f}{g}$ is continuous at a, provided $g(a) \neq 0$.

From these one obtains :

Theorem 2.13

If f and g are continuous functions, then

(i) $f + g$ is continuous

(ii) $f - g$ is continuous

(iii) cf is continuous

(iv) fg is continuous

(v) $\frac{f}{g}$ is continuous at those points where g does not take the value zero.

We use this theorem in the following examples.

$\frac{x}{|x|}$ is not continuous at 3 (Exercise 2.8[2])

Definition

A real function is said to be continuous in an (open or closed) interval if it is continuous at every point of the interval.

Explanation

When a function f is considered on a closed interval $[a, b]$, then f is said to be continuous at the end point a if $\lim\limits_{x \to a+} f(x) = f(a)$.

Similarly, f is said to be continuous at the end point b if $\lim\limits_{x \to b-} f(x) = f(b)$.

When we say that a real function f is continuous (without mentioning where) we mean that it is continuous at every point of its domain.

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Also $f(a) = \alpha$.

$\therefore$ $\lim\limits_{x \to a} f(x) = f(a)$. This means, f is continuous at a. This holds for all a in **R**. Therefore, f is continuous.

Example 2.24

The identity function is continuous.

Proof : To prove continuity of the identity function at a point a, we must prove

$$\lim_{x \to a} x = a$$

It is obvious then as $x \to a$, we have the function $x \to a$.
Thus the function x is continuous at a, for all a in **R**.
Thus it is continuous.
The following theorems will be useful to prove the continuity of functions.

Theorem 2.11

Let f and g be continuous at a. Then $f + g$ is continuous at a.

Proof. Since f is continuous at a,

$$\lim_{x \to a} f(x) = f(a).$$

Since g is continuous at a,

$$\lim_{x \to a} g(x) = g(a).$$

Adding these two,

$$\lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a).$$

But we have already seen in the previous chapter that

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

It follows that

$$\lim (f(x) + g(x)) = f(a) + g(a).$$

In other words, the function $f + g$ is continuous at a.

Corollary

If f and g are continuous, then $f + g$ is also continuous.

Restatement

Sum of two continuous functions is continuous. Similary, one can prove :

Theorem 2.12

Let f and g be two functions, continuous at a. Let c be a real number.

Then i) $f - g$ is continuous at a

ii) cf is continuous at a

iii) fg is continuous at a

iv) $\frac{f}{g}$ is continuous at a, provided $g(a) \neq 0$.

From these one obtains :

Theorem 2 13

If f and g are continuous functions, then

(i) $f + g$ is continuous

(ii) $f - g$ is continuous

(iii) cf is continuous

(iv) fg is continuous

(v) $\frac{f}{g}$ is continuous at those points where g does not take the value zero.

We use this theorem in the following examples.

Example 2.25

At what points is the function $2x+3$ continuous ?

Solution

It is continuous at every point.

The reasons are as follows :

We have already proved that the function x is continuous. Therefore, $2x$ is continuous.

We have also proved that the constant function 3 is continuous. Therefore, the function $2x+3$ is continuous.

Example 2.26

At what points is the function $\dfrac{x+1}{(x-2)(x-3)}$ continuous ?

Solution

It is not defined at the points 2 and 3. The function in the numerator, i.e., $x+1$ is continuous.

The function in the denominator $(x-2)(x-3)$ is also continuous. Therefore, the function $\dfrac{x+1}{(x-2)(x-3)}$ is continuous at all those points where the denominator does not take the value zero. Thus it is continuous at all points except at 2 and 3.

In other words, it is continuous at all points of its domain.

Remark

In the next two theorems, we prove the general results for which examples 2.25 and 2.26 become particular cases.

Theorem 2.14

Every polynomial function is continuous.

Proof : Let $a_0+a_1x+\dots+a_nx^n$ be a polynomial. We prove the continuity by induction on n.

When $n=0$, the polynomial is a_0.

It is a constant function.

Therefore, it is continuous.

When $n=1$, the polynomial is a_0+a_1x.

It is the sum of a constant function and a multiple of the identity function. Therefore, it is continuous.

Suppose, it is true that every polynomial of degree at most n is continuous.

Take a general polynomial of degree $n+1$, namely,

$a_0+a_1x+\dots+a_nx^n+a_{n+1}x^{n+1}$.

This can be written as

$a_0 + x(a_1 + a_2x + \ldots + a_nx^{n-1} + a_{n+1}x^n)$.

This is the sum of the constant function a_0 (which is continuous) and the product of the identity function x (which is continuous) and the polynomial function $a_1 + a_2x + \ldots + a_{n+1}x^n$ of degree at most n (which we assumed to be continuous).

Therefore, it is continuous.

So, continuity of a polynomial of degree n implies the continuity of a polynomial of degree $n+1$.

Thus,by the principle of induction, every polynomial function is continuous.

Theorem 2.15

Every rational function is continuous in its domain.

Proof :

Every rational function is of the form $\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions. Therefore, it is continuous at all those points where $q(x)$ is not zero. In other words, it is continuous at all points of its domain.

A method to prove continuity : Suppose we want to prove that a function f is continuous at a point a. If there is an open interval (b, c) containing a as an interior point, and a function g coinciding with f on this interval, then f is continuous at a (if and only if) g is continuous at a.

This method is used in the following :

Theorem 2.16

$|x|$ is continuous.

Proof : Let a be a real number.

Case 1 : Let $a > 0$

Consider the open interval $(0, a+1)$.a is an interior point of this interval. Also, $|x| = x$ for all x in this interval.

We already know that the identity function x is continuous. Thus, $|x|$ coincides with a continuous function on an interval containing a in the interior.

$\therefore$ $|x|$ is continuous at a.

Case 2 : Let $a < 0$.

Consider the interval $(a-1, 0)$.

a is an interior point of this interval.

Also $|x| = -x$ for all x in this interval.

We already know that the polynomial function $-x$ is continuous.

Then $|x|$ coincides with a continuous function on an interval containing a in the interior.

$\therefore$ $|x|$ is continuous at a.

Case 3 : Let $a = 0$.

Then $\lim\limits_{x \to a+} |x| = \lim\limits_{x \to 0+} x = 0$

and $\lim\limits_{x \to a-} |x| = \lim\limits_{x \to 0-} (-x) = 0,$

and $|0| = 0$.

Thus left limit at $0 =$ right limit at $0 =$ value at 0.

Therefore, $|x|$ is continuous at 0.

Conclusion

Thus in all the three cases, $|x|$ is continuous at a.

Therefore, $|x|$ is a continuous function.

Theorem 2.17

If f and g are real functions such that $f \circ g$ is defined, if g is continuous at a point a, and if f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Proof : Since g is continuous at a.

$$\lim_{x \to a} g(x) = g(a). \qquad \text{...(2.3)}$$

Since f is continuous at $g(a)$

$$\lim_{y \to g(a)} f(y) = f[g(a)].$$

Now, $\lim\limits_{x \to a} f[g(x)] = \lim\limits_{g(x) \to g(a)} f[g(x)]$ (since as $x \to a$, $g(x) \to g(a)$ because of (2.3)

$$= \lim_{y \to g(a)} f(y)$$

$$= f[g(a)].$$

Thus, $f \circ g$ is continuous at a.

Corollary

Composite of two continuous functions is continuous.

Example 2.27

Prove that the function $|1 - x + |x||$ is a continuous function.

Proof : We know that the polynomial function $1 - x$ is continuous, and that the modulus function $|x|$ is continuous. Therefore, $1 - x + |x|$ is continuous.

Let $f(x) = 1 - x + |x|$ and let $g(x) = |x|$.
Then $g(f(x)) = |f(x)| = |1 - x + |x||$.

Thus $|1 - x + |x||$ is the composite of the two continuous functions g and f. Therefore, it is continuous.

Remark

$\lim\limits_{x \to a} f(x)$ can also be written as $\lim\limits_{h \to 0} f(a+h)$ This is because, as $h \to 0$, $(a+h) \to a$. Thus $f(x)$ is continuous at a if and only if

$$\lim_{h \to 0} f(a+h) = f(a).$$

This will be used in the results that follow.

Theorem 2.18

The exponential function, e^x, is continuous.

Proof. We have proved in Theorem 2.8 that $\lim\limits_{x \to 0} e^x = 1$.

Now let a be any real number. Then

$$\begin{aligned}
\lim_{x \to a} e^x &= \lim_{h \to 0} e^{a+h} \\
&= \lim_{h \to 0} e^a . e^h \\
&= e^a . \lim_{h \to 0} e^h \\
&= e^a . 1 \\
&= e^a \\
&= \text{value of } e^x \text{ at } a.
\end{aligned}$$

This proves that e^x is continuous at every point a.

Theorem 2.19

The sine function is continuous.

Proof: We have proved in Theorem 2.3 that. $\lim\limits_{x \to 0} \sin x = 0$ and $\lim\limits_{x \to 0} \cos x = 1$.

Now, let a be any real number. Then

$$\begin{aligned}
\lim_{x \to a} \sin x &= \lim_{h \to 0} \sin (a+h) \\
&= \lim_{h \to 0} (\sin a \cos h + \cos a \sin h)
\end{aligned}$$

$$= \sin a \lim_{h\to 0} \cos h + \cos a \lim_{h\to 0} \sin h$$

$$= \sin a.1 + \cos a.0$$

$$= \sin a.$$

Thus sin x is a continuous function.

Theorem 2.20

cos x is continuous.

Remark

This can be proved similarly and therefore left as an exercise.

Remark

Later we shall prove in the next chapter that all these functions are differentiable and that all differentiable functions are continuous. That gives another method of proving the continuity.

Graphs of Continuous Functions : Even though it is not a rigorous or acceptable method, it is possible by looking at the graph of the function, in most cases, to say whether the function is continuous or not.

The graph of a continuous function has no breaks or jumps. Roughly speaking, it can be drawn without taking the pencil away from the paper.

Definition : A function that is not continuous at a point a, is said to be discontinuous at a.

Example 2.28

Let $f(x) = \begin{cases} x \text{ if } x \geq 1 \\ x^2 \text{ if } x < 1 \end{cases}$

Is f a continuous function ? Why ?

Solution

Let a be any real number. Consider three cases.

Case 1 : Let $a > 1$. Then all nearby points of a are also > 1 and, therefore, as $x \to a, f(x)$ (being equal to x) $\to a$. Thus $\lim_{x\to a} f(x) = a$ which is $f(a)$.

Case 2 : Let $a < 1$. Then all nearby points of a are also < 1 and, therefore, as $x \to a, f(x)$ (being equal to x^2) $\to a^2$. Then $\lim_{x\to a} f(x) = a^2$ which is $f(a)$.

Case 3 : Let $a = 1$. Then the nearby points can be either > 1 or < 1. We find $\lim_{x\to a+} f(x) = \lim_{x\to 1+} x = 1 = f(1)$ and $\lim_{x\to a-} f(x) = \lim_{x\to 1-} x^2 = 1 = f(1)$.

Since these two limits coincide and are equal to $f(1)$, we conclude that $\lim\limits_{x \to 1} f(x) = 1 = f(1)$.

Thus, in all the three cases $\lim\limits_{x \to a} f(x) = f(a)$. Hence, f is continuous at a.

Thus, f is continuous at all points.

Discontinuity

If a function f is not continuous at a point a, we say that f is discontinuous at a, and that a is a point of discontinuity of f.

If a is a point of discontinuity such that,

$\lim\limits_{x \to a} f(x)$ exists, then by changing the value of f at a, we can make it continuous at a. We say then that a is a removable discontinuity of f. There are other kinds of discontinuity which we do not study now.

Example 2.29

$$\text{Let } f(x) = \begin{cases} 2x \text{ if } x < 2 \\ 2 \text{ if } x = 2 \\ x^2 \text{ if } x > 2 \end{cases}$$

Show that 2 is a removable discontinuity of f.

Solution

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2-} x^2 = 2^2 = 4$$

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} 2x = 2 \times 2 = 4.$$

Thus $\lim\limits_{x \to 2} f(x) = 4$.

But $f(2) = 2 \neq 4$.

Therefore, 2 is a point of discontinuity of f.

But if we change $f(2)$ as 4 (instead of 2), then f becomes continuous.

Therefore, 2 is a removable discontinuity of f

EXERCISE 2.10

1. Prove that the function $\cos x$ is continuous.
2. Prove that the function $\tan x$ is continuous at all points of its domain.

3. Find all points at which the greatest integer function $[x]$ is continuous.

4. Prove that the function $\sin |x|$ is continuous.

Which of the following functions (questions 5 to 12) are continuous ? Why ?

5. $x + 5$

6. $f(x) = \begin{cases} x^2 \text{ if } x > 0 \\ x \text{ if } x \leq 0 \end{cases}$

7. $f(x) = \begin{cases} 2x - 1 \text{ if } x < 0 \\ 2x + 1 \text{ if } x \geq 0 \end{cases}$

8. $f(x) = \begin{cases} 2x - 1 \text{ if } x < 2 \\ \dfrac{3x}{2} \text{ if } x \geq 2 \end{cases}$

9. $f(x) = \begin{cases} \sin x \text{ if } x < 0 \\ x \text{ if } x \geq 0 \end{cases}$

10. $f(x) = \begin{cases} \dfrac{\sin x}{x} \text{ if } x < 0 \\ x + 1 \text{ if } x \geq 0 \end{cases}$

11. $f(x) = \begin{cases} x \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}$

12. $f(x) = \begin{cases} \dfrac{x}{|x|} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$

13. Assuming that $\lim\limits_{x \to 1} \log x = 0$

and using the formula $\log x = \log a + (\log \frac{x}{a})$ for all $a > 0$, prove that $\log x$ is a continuous function.

14. Locate the points of discontinuity of the following functions :

(i) $f(x) = \begin{cases} \dfrac{x^4 - 16}{x - 2} \text{ if } x \neq 2 \\ 16 \text{ if } x = 2 \end{cases}$

(ii) $f(x) = \begin{cases} x^3 - x^2 + 2x - 2 \text{ if } x \neq 1 \\ 4 \text{ if } x = 1 \end{cases}$

15. In the following, determine the value of the constant so that the given function is continuous :

(i) $f(x) = \begin{cases} kx^2 \text{ if } x \leq 2 \\ 3 \text{ if } x > 2 \end{cases}$

(ii) $g(x) = \begin{cases} ax + 5 \text{ if } x \leq 2 \\ x - 1 \text{ if } x > 2 \end{cases}$

(iii) $h(x) = \begin{cases} m(x^2 - 2x) \text{ if } x < 0 \\ \cos x \text{ if } x \geq 0 \end{cases}$

16. If $f(x) = \begin{cases} 1 \text{ if } x \leq 3 \\ ax + b \text{ if } 3 < x < 5 \\ 7 \text{ if } 5 \leq x, \end{cases}$

determine the values of a and b, so that $f(x)$ is continuous.

CHAPTER 3

DIFFERENTIATION

3.1 Derivative of a Function

IN THIS CHAPTER, we consider real functions defined on either **R** or an open interval (a, b) or (a, ∞) or $(-\infty, a)$.

Definition . Let a be a point in the domain of the real function f.
Then

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if it exists, is called the *derivative of f at a.*

Motivation : The derivative of f at a, measures the rate at which the value of f changes, for slight changes in the variable, at a.

Let x be the variable.
Let it be given a small increment Δx.
(This Δx may be either positive or negative)
Then the variable becomes $x + \Delta x$.
The value of f also changes from $f(x)$ to $f(x + \Delta x)$.
The difference in the value of f is $f(x + \Delta x) - f(x)$.
This change has taken place for a change Δx in the variable x.

Therefore, the rate of change is $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$.

When x is fixed as a, and when Δx is very small, what is this rate of change ?
In our notation, this question can be reformed :

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ is what when } x = a\ ?$$

In other words, what is $\lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ at $x = a$? This limit, when it exists, is the rate of change of f at a. Replacing Δx by h, this is the same as $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$. This is what we define as the derivative of f at a.

$$\text{Derivative of } f \text{ at } a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Physical Meaning

To make the above motivation clearer, imagine that the variable x is the time x, of a moving particle. We attempt now a reasonable definition of the speed at a particular time a. We are familiar with the notion of speed in a time-interval. It is given by

$$\text{Speed} = \frac{\text{Distance travelled}}{\text{time taken}}.$$

If the interval is a small one containing a, we expect that the speed in the interval, may be taken as an approximation of the speed at time a. If the time interval is between a and $a + h$, (h positive or negative) then the speed in that duration is

$$\frac{\text{Distance}}{\text{Time}} = \frac{\text{Distance travelled upto time}(a+h) - \text{Distance travelled upto time } a}{\text{Length of the time interval}}$$

$$= \frac{f(a+h) - f(a)}{h}.$$

As $h \to 0$, this quantity $\to$ the speed at time a. Thus, the derivative is related to a function, in the same way as velocity is related to the distance travelled by a moving particle.

Geometrical meaning

Let $y = f(x)$ be a function and a be a point in the domain. Then draw the graph of this function. It is a curve (Fig. 3.1). $(a, f(a))$ is a point P on this curve. Take a nearby point $a + h$ in the domain. (This can be on either side of a, since h can be positive or negative). Correspondingly $(a+h, f(a+h))$ is a point Q on the graph. Join these two points by a segment. This is a chord of the curve.

The slope of this chord PQ is equal to $\frac{QR}{PR}$.

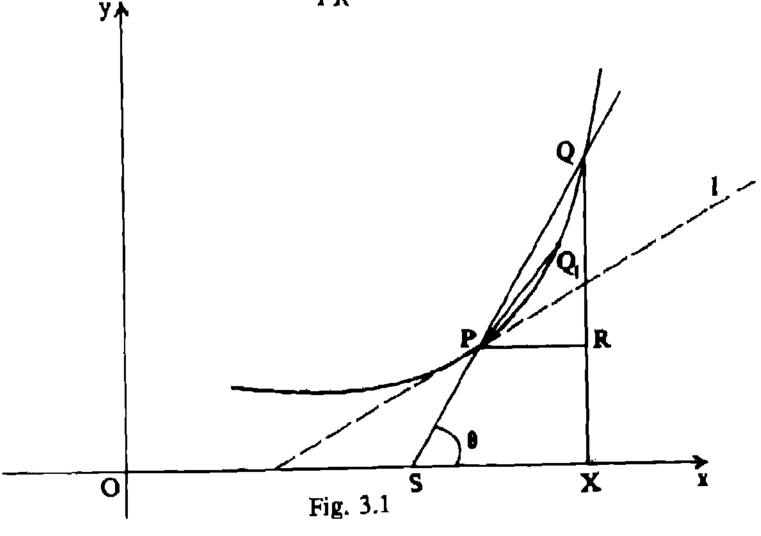


Fig. 3.1

[*Explanation* . Extend QP to meet the x-axis at S.
Let θ be the angle PSX The slope of PQ is $\tan\theta$. Draw a line through P parallel to the x-axis and a line meet at R Then the angle QPR is also equal to θ. Therefore, the slope of $PQ = \tan QPR = \frac{QR}{PR}$.]

Now, QR = difference in the y-coordinates of Q and P
$= f(a+h) - f(a)$.
and PR = difference in the x-coordinates of Q and P
$= (a+h) - a = h$.

Thus the slope of PQ is $= \dfrac{f(a+h) - f(a)}{h}$.

As Q comes nearer and nearer to P along the curve, the chord QP approaches the tangent at P [*Explanation* : As Q approaches P along the curve, QP taking the intermediate positions like QP, finally approaches the tangent l at P. This happens when $h \to 0$].

$$\text{Therefore,} \quad \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{Q\to P} (\text{slope of } PQ)$$
$$= \text{slope of the tangent at } P.$$

Thus the derivative of f at a is the slope of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$.

Equivalent Notations and Terms

We denote the derivative of f at a by $f'(a)$. When the derivative exists for every a in the domain, then it associates a number to each point in the domain of f and thus it is a real function. We denote it by f'. We say that f is differentiable at a if $f'(a)$ exists. We say that f is differentiable if f' exists.

The derivative is sometimes called also as differential coefficient. But we shall use the word derivative only in this book.

Sometimes, the function is written in the notation $y = f(x)$, as for example, $y = mx + c$ or $y = x^2$. Then the derivative function is denoted by $\frac{dy}{dx}$ or by $\frac{d}{dx}[f(x)]$ or by $Df(x)$ or by y'. Note that here $\frac{dy}{dx}$ is just a symbol for the derivative, it is not the quotient of two quantities dy and dx. The process of taking the derivative is known as differentiation.

Theorem 3.1

The derivative of x^n is nx^{n-1}, where n is a fixed number, integer or rational.

Proof : Let $y = x^n$ Then by definition

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = \lim_{x+\Delta x\to x} \frac{(x+\Delta x)^n - x^n}{(x+\Delta x) - x}$$

This limit exists and is equal to nx^{n-1}, because of the formula $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$ proved earlier

Therefore, $\dfrac{dy}{dx} = nx^{n-1}$

or $\dfrac{d}{dx}(x^n) = n\,x^{n-1}$.

Example 3.1

When $n = 3$, $y = x^3$ and $\dfrac{dy}{dx} = 3x^{3-1} = 3x^2$ or $\dfrac{d}{dx}(x^3) = 3x^2$

When $n = \dfrac{1}{2}$, $y = \sqrt{x}$ and $\dfrac{dy}{dx} = \dfrac{1}{2}x^{\frac{1}{2}-1} = \dfrac{1}{2\sqrt{x}}$

or $\dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$.

Theorem 3.2

The derivative of the constant function $f(x) = c$ is the zero function.

Proof :

Let $f(x) = c$ $\therefore f(x+h)$ is also c

$\therefore f(x+h) - f(x) = 0$.

$$\because f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 0 = 0.$$

Thus the derivative of the constant function is the constant function 0.

Theorem 3.3

If a function f is differentiable at a point a, then it is continuous at that point.

Proof : By assumption, $f'(a)$ exists. Now,

$$\lim_{h \to 0} [f(a+h) - f(a)] = \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \quad h \right]$$

$$= \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \right] \lim_{h \to 0} h$$

$$= f'(a) \,.\, 0$$

$$= 0$$

It follows that $\lim\limits_{h \to 0} f(a+h) = f(a)$ or equivalently, $\lim\limits_{x \to a} f(x) = f(a)$.

Therefore, f is continuous at a.

Corollary

Every differentiable function is continuous.

Remark

However, not every continuons function is differentiable.
For example, consider the function $f(x) = |x|$
We have already proved that this function is continuous. Let us examine whether $f'(0)$ exists.

Now,
$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{|h| - 0}{h}$$
$$= \lim_{h \to 0} \frac{|h|}{h}$$

Now, as you can easily see,

$\frac{|h|}{h} = 1$ if h is positive and $= -1$ if h is negative

$$\therefore \quad \lim_{h \to 0+} \frac{|h|}{h} \neq \lim_{h \to 0-} \frac{|h|}{h}$$

Hence,
$$\lim_{h \to 0} \frac{|h|}{h} \text{ does not exist}$$

That is,
$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \text{ does not exist.}$$

In other words, $f'(0)$ does not exist.
So, though the function $|x|$ is continuous at 0, it is not differentiable at 0.

Differentiation from first principle

When the derivative of a function is calculated directly using the definition of the derivative, it is called differentiation from first principle.

The following examples illustrate the method of finding the derivatives from first principle.

Example 3 2

Find the derivative of the function $y = x^{-\frac{3}{2}}$ from first principle .

Solution

By definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^{-\frac{3}{2}} - x^{-\frac{3}{2}}}{h}$$

$$= \lim_{h\to 0} \frac{x^{-\frac{3}{2}}\left(1+\frac{h}{x}\right)^{-\frac{3}{2}} - x^{-\frac{3}{2}}}{h}$$

$$= \lim_{h\to 0} \frac{x^{-\frac{3}{2}} \left[1 + \left(-\frac{3}{2}\right)\frac{h}{x} + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)\left(\frac{h}{x}\right)^2}{1.2} + \dots \right] - x^{-\frac{3}{2}}}{h}$$

(Binomial theorem for a rational index)

$$= \lim_{h\to 0} \left[\left(\frac{-3}{2}\right) \frac{x^{-\frac{3}{2}}}{x} + \text{terms containing } h \right]$$

$$= \lim_{h\to 0} \left[\frac{-3}{2} . x^{-\frac{3}{2}-1} \right] + \lim_{h\to 0} \left[\text{terms containing } h\right]$$

$$= -\frac{3}{2} x^{-\frac{5}{2}} + 0 = -\frac{3}{2} x^{-\frac{5}{2}}$$

$$\therefore \frac{d}{dx}\left(x^{-\frac{3}{2}}\right) = -\frac{3}{2} x^{-\frac{5}{2}}$$

Example 3 3

Find the derivative of the function $f(x) = 3x^2 + 5x - 1$.

Solution

By definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[3(x+h)^2 + 5(x+h) - 1] - [3x^2 + 5x - 1]}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 + 5h}{h}$$

$$= \lim_{h \to 0} (6x + 5 + 3h)$$

$$= \lim_{h \to 0} (6x + 5) + \lim_{h \to 0} (3h)$$

$$= 6x + 5 + 0$$

$$= 6x + 5.$$

$$\therefore \frac{d}{dx}(3x^2 + 5x - 1) = 6x + 5.$$

EXERCISE 3.1

1. If $y = 2x$, find $\frac{dy}{dx}$ from the first principle.
2. If $f(x) = (x - 1)^2$, find $f'(x)$ from the first principle.
3. If $f(x) = x^n$ and if $f'(1) = 10$, find the value of n.
4. Prove from first principle that $\frac{d}{dx}(\alpha x^n) = \alpha n\, x^{n-1}$.
5. If $f(x) = \alpha x^n$, prove that $\alpha = \frac{f'(1)}{n}$
6. Prove that the greatest integer function $[x]$ is not differentiable at $x = 1$. (Hint : It is not continuous)
7. If $y = ax + b$, find $\frac{dy}{dx}$ from the first principle.
8. If $f(x) = mx + c$ and if $f(0) = f'(0) = 1$. What is $f(2)$?
9. If $f(x) = \frac{2}{x}$, find $f'(x)$ from first principle.
10. If $f(x) = \frac{x + 2}{3x + 5}$ find $f'(x)$ from first principle.

3.2 Differentiation of Some Important Functions

In this section, we find the derivatives of polynomial functions, trigonometric functions, the exponential function, the logarithmic function and some other related functions.

Theorem 3.4

$$\frac{d}{dx}(\sin x) = \cos x.$$

Proof : Let $f(x) = \sin x$.

Then $f(x+h) = \sin(x+h)$.

$$\therefore f(x+h) - f(x) = \sin(x+h) - \sin x$$

$$= 2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)$$

$$= 2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right).$$

$$\therefore \frac{f(x+h)-f(x)}{h} = \frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}$$

$$= \cos\left(x+\frac{h}{2}\right) \times \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$\therefore \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{\frac{h}{2}\to 0} \cos\left(x+\frac{h}{2}\right) \lim_{\frac{h}{2}\to 0} \frac{\left(\sin\frac{h}{2}\right)}{\frac{h}{2}}$$

$$= \cos x.1$$

$$= \cos x.$$

$$\therefore \frac{d}{dx}(\sin x) = \cos x.$$

Remark

Here we have used the result , $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ (which is already proved) and

$$\lim_{h\to 0} \cos(x+h) = \cos x.$$

This latter result is proved as follows :

$$\begin{aligned}\lim_{h\to 0} \cos(x+h) &= \lim_{h\to 0} (\cos x \cos h - \sin x \sin h)\\ &= \cos x \lim_{h\to 0} \cos h - \sin x \lim_{h\to 0} \sin h\\ &= \cos x.1 - \sin x.0\\ &= \cos x.\end{aligned}$$

Remark

Similarly, one can prove $\frac{d}{dx}(\cos x) = -\sin x$,

Proof is left to the students as an exercise.

Theorem 3.5

$$\frac{d}{dx}(e^x) = e^x.$$

Proof : Let $f(x) = e^x$

Then $f(x+h) = e^{x+h} = e^x\ e^h$

$$\begin{aligned}\therefore \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{e^x.e^h - e^x}{h}\\ &= \lim_{h\to 0} e^x \left(\frac{e^h - 1}{h}\right)\\ &= e^x. \lim_{h\to 0} \frac{e^h - 1}{h}\\ &= e^x.1 \text{ (because we know } \lim_{h\to 0} \frac{e^h-1}{h} = 1\text{)}\\ &= e^x.\end{aligned}$$

Thus $\frac{d}{dx}(e^x) = e^x$.

Remark

e^x is the derivative of itself.

Theorem 3.6

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

Proof . Let $f(x) = \tan x$.

Then $f(x+h) - f(x) = \tan(x+h) - \tan x$

$$= \frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{\cos(x+h)\cos x}$$

$$= \frac{\sin(x+h-x)}{\cos(x+h)\cos x} = \frac{\sin h}{\cos(x+h)\cos x}.$$

$$\therefore \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin h}{h.\cos(x+h).\cos x}$$

$$= \lim_{h\to 0}\left[\frac{\sin h}{h} . \frac{1}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to 0}\left(\frac{\sin h}{h}\right) . \lim_{h\to 0} \frac{1}{\cos(x+h).\cos x}$$

$$= 1. \lim_{h\to 0} \frac{1}{\cos(x+h).\cos x}$$

$$= 1. \frac{1}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x.$$

Thus $\frac{d}{dx}(\tan x) = \sec^2 x.$

Remark

This can also be proved by a slightly different method given below :

Second Proof : Let $f(x) = \tan x$.

Then $f(x+h) - f(x) = \tan(x+h) - \tan x$

$$= \frac{\tan x + \tan h}{1 - \tan x.\tan h} - \tan x = \frac{(\tan x + \tan h) - (\tan x - \tan^2 x \tan h)}{1 - \tan x.\tan h}$$

$$= \frac{\tan h(1 + \tan^2 x)}{1 - \tan x.\tan h}$$

$$= \frac{\tan h.\sec^2 x}{1 - \tan x.\tan h}$$

$$= \frac{\sin h \sec^2 x}{\cos h - \tan x \sin h}$$

$$\therefore \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \left[\frac{\sin h}{h} \, . \sec^2 x . \frac{1}{\cos h - \tan x . \sin h} \right]$$

$$= \lim_{h\to 0} \left[\frac{\sin h}{h} \sec^2 x \right] . \frac{1}{\lim\limits_{h\to 0} [\cos h - \tan x . \sin h]}$$

$$= 1 . \sec^2 x . \frac{1}{1 - \tan x \, 0} \quad \left(\because \lim_{h\to 0} \frac{\sin h}{h} = 1 \right)$$

$$= \sec^2 x$$

Thus $\frac{d}{dx}(\tan x) = \sec^2 x.$

Theorem 3.7

$\frac{d}{dx}(\log x) = \frac{1}{x}.$

Proof : Let $f(x) = \log x$

Then $f(x+h) - f(x) = \log(x+h) - \log x$

$$= \log \left(\frac{x+h}{x} \right)$$

$$= \log \left(1 + \frac{h}{x} \right)$$

$$\therefore \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\log \left(1 + \frac{h}{x} \right)}{h}$$

$$= \lim_{h\to 0} \left[\frac{\log(1+\frac{h}{x})}{\frac{h}{x}} \, . \frac{1}{x} \right]$$

$$= \lim_{y\to 0} \left[\frac{\log(1+y)}{y} \right] . \lim_{h\to 0} \frac{1}{x}, \text{where } y = \frac{h}{x}$$

(As $h \to 0, y \to 0$)

$$= 1.\frac{1}{x} \quad \left(\text{because we know } \lim_{y\to 0}\frac{\log(1+y)}{y} = 1\right)$$

$$= \frac{1}{x}.$$

Theorem 3.8

The derivative of $f+g$ is equal to $f'+g'$. Stated in words the theorem is : derivative of the sum = sum of the derivatives.

Or to be precise, if f and g are differentiable, then $f+g$ is differentiable and $(f+g)'=f'+g'$.

Proof :

$$\frac{(f+g)(x+h)-(f+g)(x)}{h}$$

$$= \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$$

$$= \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}$$

$$\therefore\ (f+g)'(x) = \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= f'(x)+g'(x).$$

Theorem 3.9

Let f be a differentiable function and let c be a fixed real number. Then $(cf)'=cf'$.

Proof : $$(cf)'(x) = \lim_{h\to 0}\frac{(cf)(x+h)-(cf)(x)}{h}$$

$$= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$

$$= c.\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= c\ f'(x).$$

Corollary

If f and g are differentiable functions, then $f-g$ is also differentiable and

$$(f-g)'=f'-g'.$$

Proof : $(f-g)' = (f+(-g))' = f' + (-g)'$

$$= f' + [(-1)g]' = f' + (-1)g' = f' - g'.$$

Example 3.4

Differentiate $4x^3 - 9 - 6x^2$.

Solution

We know $\frac{d}{dx}(4x^3) = 4.\frac{d}{dx}(x^3) = 4(3x^2) = 12x$.

$$\frac{d}{dx}(-9) = 0.$$

$$\frac{d}{dx}(-6x^2) = -6\frac{d}{dx}(x^2) = -6(2x) = -12x.$$

$$\therefore \frac{d}{dx}(4x^3 - 9 - 6x^2) = 12x^2 + 0 - 12x$$

$$= 12x^2 - 12x.$$

Example 3.5

Differentiate $5 \sin x - 2 \log x$.

Solution

$$\frac{d}{dx}(5 \sin x - 2 \log x) = \frac{d}{dx}(5 \sin x) - \frac{d}{dx}(2 \log x)$$

$$= 5\frac{d}{dx}(\sin x) - 2\frac{d}{dx}(\log x)$$

$$= 5 \cos x - \frac{2}{x}.$$

EXERCISE 3.2

1. Differentiate $\cos 3x$ from first principle.
2. Differentiate $\cot(2x+1)$ from first principle.

Differentiate the following functions :

3. ax^3+bx^2+cx+d

4. $x+\dfrac{1}{x}$

5. $(x-1)(x-2)$

6. $e^x+2\cos x$

7. $x^{\frac{2}{3}}+2x^2$

8. $\dfrac{(x^2+1)(x+3)}{x}$

9. $\dfrac{1}{3}e^x-5e$

10. $(x+\dfrac{1}{x})^2$

11. $\tan x+2\sin x+3\cos x-\dfrac{1}{2}\log x-e^x$

12. $\log x^2$

3.3 Product Rule of Differentiation

In this section we prove the formula

$$(fg)'=fg'+f'g$$

and apply it to differentiate products of functions.

Theorem 3.10

Let f and g be differentiable functions. Then their product fg is also differentiable and

$$(fg)'=fg'+f'g.$$

Proof :

$$\frac{(fg)(x+h)-(fg)(x)}{h}=\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$=\frac{f(x+h)\,g(x+h)-f(x+h)\,g(x)+f(x+h)\,g(x)-f(x)\,g(x)}{h}$$

$$=f(x+h)\left(\frac{g(x+h)-g(x)}{h}\right)+g(x)\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$\therefore \lim_{h\to 0} \frac{(fg)(x+h)-(fg)(x)}{h}$$

$$= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + g(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= f(x)\, g'(x) + g(x) f'(x).$$

Here, $\lim f(x+h) = f(x)$ because f is continuous at x, f being differentiable at x.

Thus $(fg)'(x)$ exists and equals $(fg' + g'f)(x)$.

Remark

The rule $(fg)' = fg' + gf'$ is called the product rule of differentiation. It can also be written as

$$\frac{d}{dx}(fg). = f\frac{dg}{dx} + g\frac{df}{dx}.$$

It is also remembered as follows :

Derivative of the product of two functions
= (first function). derivative of second function
+ (second function). derivative of first function.

There is another form of this rule which is easy to remember. It is the following :

$$\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}.$$

This result which is proved for the product of two functions can be extended for the product of any finite number of functions and the rule is

$$\frac{(fgh\ldots)'}{fgh\ldots} = \frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h} + \ldots$$

Example 3.6

Differentiate $x\, e^x$.

Solution

$$\frac{d}{dx}(xe^x) = x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$$

$$= xe^x + e^x.1$$

$$= e^x(x+1).$$

Example 3.7

Differentiate $(x-1)(x-2)$ using product rule. Differentiate the same after expanding as a polynomial. Verify that the two answers are the same.

Solution

$$\begin{aligned}\frac{d}{dx}[(x-1)(x-2)] &= (x-1)\frac{d}{dx}(x-2)+(x-2)\frac{d}{dx}(x-1)\\ &= (x-1)\cdot(1)+(x-2)(1)\\ &= x-1+x-2\\ &= 2x-3.\end{aligned}$$

Also, $(x-1)(x-2) = x^2-3x+2$,

$$\begin{aligned}\therefore \frac{d}{dx}[(x-1)(x-2)] &= \frac{d}{dx}(x^2)-3\frac{d}{dx}(x)+\frac{d}{dx}(2)\\ &= 2x-3.\end{aligned}$$

We note that both the methods give the same answer.

Example 3.8

Differentiate $x^2e^x \sin x$.

Solution

First we differentiate x^2e^x :

$$\begin{aligned}\frac{d}{dx}(x^2e^x) &= x^2\frac{d}{dx}(e^x)+e^x\frac{d}{dx}(x^2)\\ &= x^2e^x+2xe^x=(x^2+2x)e^x.\end{aligned}$$

$$\begin{aligned}\text{Now, }\frac{d}{dx}(x^2e^x \sin x) &= x^2e^x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(x^2e^x)\\ &= x^2e^x\cos x+\sin x\,(x^2+2x)\,e^x\\ &= e^x(x^2\cos x+x^2\sin x+2x\sin x)\\ &= xe^x(x\cos x+x\sin x+2\sin x).\end{aligned}$$

This can be done also as follows :

$$\begin{aligned}\frac{\frac{d}{dx}(x^2e^x\sin x)}{x^2e^x\sin x} &= \frac{\frac{d}{dx}(x^2)}{x^2}+\frac{\frac{d}{dx}(e^x)}{e^x}+\frac{\frac{d}{dx}(\sin x)}{\sin x}\\ &= \frac{2x}{x^2}+\frac{e^x}{e^x}+\frac{\cos x}{\sin x}\\ &= \frac{2}{x}+1+\frac{\cos x}{\sin x}.\end{aligned}$$

$$\begin{aligned}\therefore \frac{d}{dx}(x^2e^x\sin x) &= 2x\,e^x\sin x+x^2e^x\sin x+x^2e^x\cos x\\ &= x\,e^x(2\sin x+x\sin x+x\cos x).\end{aligned}$$

EXERCISE 3.3

Differentiate the following functions (1 to 10) .

1. $x \sin x$
2. $x \sin x \log x$
3. $x^{-\frac{1}{3}} e^x$
4. $\sec x \tan x$
5. $(1+x^2) \cos x$
6. $e^x \cot x$
7. $x \operatorname{cosec} x$
8. $(x^2-4x+5)(x^3-2)$
9. $(1-2 \tan x)(5+4 \sin x)$
10. $\sin^2 x$
11. Differentiate x^2 by product rule and verify that the answer is $2x$.
12. Differentiate $(x^2+ax+b)(x^2+cx+d)$ in two ways, first by product rule, and then by expanding the product. Verify that the two answers are same.
13. Differentiate in two ways, using product rule and otherwise, the function $(1 + 2 \tan x)(5 + 4 \cos x)$. Verify that the answers are the same.
14. We know that $\tan x \cos x = \sin x$ and $\frac{d}{dx}(\sin x) = \cos x$, Differentiate $\tan x \cos x$ by product rule and check that the answer is $\cos x$.
15. If u, v, w are three differentiable functions. Prove that $$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}.$$
16. Using the formula in problem 15, differentiate $x \sin x\, e^x$.

3.4 Quotient Rule for Differentiation

In this section, we prove the formula

$$\left(\frac{f}{g}\right)' = \frac{gf'-fg'}{g^2}$$

and use it to differentiate quotients of functions.

Theorem 3.11

Let f and g be differentiable functions. Then $\dfrac{f}{g}$ is differentiable at all points where g does not take the value zero, and

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x)-f(x)g'(x)}{[\,g(x)\,]^2}$$

Remark

It is good to remember this in the following form :
Derivative of a quotient

$$= \frac{(\text{denominator})\,(\text{derivative of numerator}) - (\text{numerator})\,(\text{derivative of denominator})}{(\text{denominator})^2}.$$

We give two proofs of this theorem.

First Proof : We have

$$\left(\frac{f(x)}{g(x)}\right)'$$

$$= \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} \qquad \left(\text{by the meaning of } \frac{f}{g}\right)$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{h\,g(x+h)\,g(x)}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h\,g(x+h)g(x)}$$

$$= \lim_{h\to 0}\frac{\dfrac{f(x+h)-f(x)}{h}\,.\,g(x) - \dfrac{g(x+h)-g(x)}{h}\,.\,f(x)}{g(x+h)\,.\,g(x)}$$

$$= \frac{g(x)\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} - f(x)\displaystyle\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}}{g(x)\displaystyle\lim_{h\to 0} g(x+h)}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)g(x)}.$$ [Here we use $\lim_{h\to 0} g(x+h) = g(x)$,

because g is given to be differentiable and hence continuous]

$$= \frac{g(x)\, f'(x) - f(x)\; g'(x)}{[\, g(x)\,]^2} \,.$$

Second Proof . First we shall prove that

$$\left(\frac{1}{g(x)} \right)' = -\frac{g'(x)}{[\, g(x)\,]^2}$$

Now,

$$\left(\frac{1}{g(x)} \right)' = \lim_{h \to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{g(x) - g(x+h)}{h\; g(x+h)\; g(x)}$$

$$= \lim_{h \to 0} \frac{-[\, g(x+h) - g(x)\,]}{h} \cdot \frac{1}{g(x+h)} \cdot \frac{1}{g(x)}$$

$$= -g'(x) \cdot \frac{1}{\lim\limits_{h \to 0} g(x+h)} \cdot \frac{1}{g(x)}$$

$$= \frac{-g'(x)}{g(x) \cdot g(x)} \qquad [\text{because } \lim_{h \to 0} g(x+h) = g(x)]$$

$$= \frac{-g'(x)}{[\, g(x)\,]^2} \,.$$

Our next step is to consider $\frac{f(x)}{g(x)}$ as the product of $f(x)$ and $\frac{1}{g(x)}$ and apply the product rule of differentiation.

Thus,

$$\left(\frac{f(x)}{g(x)} \right)' = f(x) \left(\frac{1}{g(x)} \right)' + \frac{1}{g(x)} \cdot f'(x)$$

$$= f(x) \left(-\frac{g'(x)}{[\, g(x)\,]^2} \right) + \frac{1}{g(x)} \cdot f'(x)$$

$$= \frac{1}{(g(x))^2} \left[\; f(x) . \left(-g'(x)\right) + g(x)\, f'(x) \;\right]$$

$$= \frac{g(x)\, f'(x) - f(x)\, g'(x)}{(g(x))^2}.$$

Example 3.9

Differentiate $\dfrac{2x+3}{x^2-5}$.

Solution

Let $f(x) = 2x+3$ and $g(x) = x^2-5$. Thus $f'(x) = 2$ and $g'(x) = 2x$.

$$\therefore \frac{d}{dx}\left(\frac{2x+3}{x^2-5}\right) = \frac{g(x)f'(x) - f(x)\, g'(x)}{(g(x))^2}$$

$$= \frac{(x^2-5)(2) - (2x+3)(2x)}{(x^2-5)^2}$$

$$= \frac{-2x^2-6x-10}{(x^2-5)^2}.$$

Example 3.10

Differentiate $\dfrac{e^x}{1+\sin x}$.

Solution

$$\frac{d}{dx}\left(\frac{e^x}{1+\sin x}\right) = \frac{(1+\sin x)\dfrac{d}{dx}(e^x) - e^x \dfrac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)\, e^x - e^x \cos x}{(1+\sin x)^2}$$

$$= \frac{e^x (1+\sin x - \cos x)}{(1+\sin x)^2}.$$

Example 3.11

We know that $\tan x = \dfrac{\sin x}{\cos x}$ and $\dfrac{d}{dx}(\tan x) = \sec^2 x$.

Use quotient rule for differentiating $\dfrac{\sin x}{\cos x}$ and verify that the answer is $\sec^2 x$.

Solution

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} \qquad (\text{because } \cos^2 x + \sin^2 x = 1)$$

$$= \sec^2 x$$

EXERCISE 3.4

Differentiate the following functions (1 to 8) :

1. $\dfrac{x+\sin x}{x+\cos x}$
2. $\dfrac{x}{1+\tan x}$
3. $\dfrac{\log x}{x}$
4. $\dfrac{e^x}{x}$
5. $\dfrac{ax^2+bx+c}{px^2+qx+r}$
6. $\dfrac{1}{ax^2+bx+c}$
7. $\dfrac{\sin x+\cos x}{\sin x-\cos x}$
8. $\dfrac{\sec x-1}{\sec x+1}$

9. We know that $\dfrac{\sec x}{\operatorname{cosec} x} = \tan x$ and that $\dfrac{d}{dx}(\tan x) = \sec^2 x$

Differentiate $\dfrac{\sec x}{\operatorname{cosec} x}$ by quotient rule and verify that the answer is the same.

10. We know that $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{d}{dx}(x^{-1}) = -x^{-2}$.

Differentiate $\dfrac{1}{x}$ by quotient rule and verify that the answer is the same

3.5 Differentiation of a Function of a Function

In this section we give a rule for differentiating the composite of two functions. The formula is

$$[f(g(x))]' = f'(g(x))\, g'(x).$$

This is useful to differentiate a variety of functions.

Theorem 3.12

If f and g are differentiable functions (wherever it is defined), then $f \circ g$ is also differentiable and $(f\circ g)'(x) = f'(g(x))\, g'(x)$.

Proof : We shall be using the following fact. If a function f is differentiable at x, then

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

and, therefore, $\dfrac{f(x+h)-f(x)}{h} = f'(x) +$ a function of h whose limit as $h \to 0$ is zero.

If this function of h is named $f_1(h)$, we have,

$$f(x+h)-f(x) = h[f'(x)+f_1(h)] \text{ and } \lim_{h\to 0} f_1(h) = 0 \qquad \ldots(3.1)$$

Now,

$$(f\circ g)'(x) = \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}$$

$$= \lim_{h\to 0}\frac{f(y+k)-f(y)}{h}, \text{ where } y = g(x) \text{ and } k = g(x+h) - g(x)$$

$$= \lim_{h\to 0}\left[\frac{k}{h}\cdot\frac{f(y+k)-f(y)}{k}\right]$$

$$= \lim_{h\to 0}\left[\frac{k}{h}\left(f'(y)+f_1(k)\right)\right], \text{ where } \lim_{k\to 0} f'_1(k)=0 \text{ by } (3.1)$$

$$= \lim_{h\to 0}\frac{k}{h}f'(y)+\lim_{\substack{h\to 0\\ k\to 0}}\frac{k}{h}f_1(k)$$

$$= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}f'(y)+\lim_{h\to 0}\frac{g(x+h)-g(x)}{\mathrm{h}}\cdot\lim_{h\to 0}\frac{k}{h}f_1(k)$$

$$= g'(x).f'(y)+g'(x).0$$

$$= g'(x).f'(y)=g'(x)f'(g(x))=f'(g(x)).g'(x).$$

Remark

This rule is known as chain rule or function of a function rule. It can be restated as follows : Put $g(x)=y$. Then, $f(g(x))=f(y)$.

We want to differentiate $f(y)$ with respect to x.

The rule says : First differentiate with respect to y.

One gets $f'(y)$ which is same as $f'(g(x))$.

Then take $\frac{dy}{dx}$ which is same as $g'(x)$.

Multiply the two and get $f'(g(x))\,g'(x)$.

This is what $(f \circ g)'(x)$ is.

In other words, if $y=g(x)$ and $z=f(y)$.

Then $\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}$.

This is easy to remember. We see this as if we are cancelling the dy occurring both in the numerator and the denominator on the right side. But note that dy does not have a separate meaning so for. It is only a part of a meaningful symbol for the derivative.

You can remember the chain rule in this way.

Derivative of z w.r.t. x = Derivative of z w.r.t. y × Derivative of y w.r.t. x

This chain rule can be extended. For example,

Derivative of z w.r.t. x

= Derivative of z w.r.t u × Derivative of u w.r.t. v × Derivative of v w.r.t. x.

Example 3.12

Differentiate $\sin x^2$.

Solution

Put $y = x^2$ and $z = \sin y$. Then $\dfrac{dy}{dx} = 2x$ and $\dfrac{dz}{dy} = \cos y$.

$$\therefore \frac{d}{dx}(\sin x^2) = \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = (\cos y)(2x)$$

$$= (\cos x^2)(2x) = 2x \cos x^2.$$

This solution can be rewritten using a more convenient notation in the following manner :

$$\frac{d}{dx}(\sin x^2) = \frac{d(\sin x^2)}{d(x^2)} \cdot \frac{d(x^2)}{dx} = \cos x^2 \ 2x = 2x \cos x^2.$$

Example 3.13

Differentiate $e^{\sin x}$.

Solution

Put $y = \sin x$ and $z = e^y$.

Then $\frac{dy}{dx} = \cos x$ and $\frac{dz}{dy} = e^y$.

$$\frac{d}{dx}(e^{\sin x}) = \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = e^y. \cos x = e^{\sin x}. \cos x.$$

We may also write in this way .

$$\frac{d(e^{\sin x})}{dx} = \frac{d(e^{\sin x})}{d(\sin x)} \cdot \frac{d(\sin x)}{dx}$$

$$= e^{\sin x}. \cos x.$$

Example 3.14

The function $\sin^2 x$ is the product $\sin x \sin x$ and therefore can be differentiated by product rule. It is also the composite $f \circ g$ of the function $f(x) = x^2$ and the function $g(x) = \sin x$. So it can be differentiated by the chain rule. Verify that the answers are same.

Solution

$$\frac{d}{dx}(\sin^2 x) = \frac{d}{dx}(\sin x \sin x)$$

$$= \sin x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(\sin x)$$

$$= \sin x \cos x + \sin x \cos x$$

$$= 2 \sin x \cos x.$$

Also, let $y = \sin x$ and $z = y^2$.

$$\text{Then } \frac{d}{dx}(\sin^2 x) = \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = 2y.\cos x$$

$$= 2 \sin x \cos x.$$

Thus, both methods give the same answer

EXERCISE 3.5

Differentiate the following functions (1 to 8) :

1. $\cos x^2$
2. $e^{\cos x}$
3. $\sin 2x + (2x-5)^2$
4. $\sin 2x \cos 3x$
5. $\dfrac{x}{\sin 3x}$
6. $\log \sin x$
7. $\dfrac{1}{\log \cos x}$
8. e^{x^2}
9. We know that $e^{\log x} = x$ Differentiate $e^{\log x}$ by chain rule and verify that the answer is 1.
10. We know that $\sin 2x = 2 \sin x \cos x$. Differentiate $\sin 2x$ by chain rule, differentiate $2 \sin x \cos x$ by product rule, and verify that the answers are same.

3.6 Differentiation of Inverse Trigonometric Functions

In this section you will learn to find the derivative of $\sin^{-1}x$, $\tan^{-1}x$ etc. This method can be used to find the derivative of any inverse function

Example 3.15

Differentiate $\sin^{-1}x$

Solution

Let $y = \sin^{-1}x$.
Then $\sin y = x$.

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$\text{i.e.}\ \frac{d(\sin y)}{dy}\ \frac{dy}{dx} = \frac{d}{dx}(x)$$

$$\text{i.e.}\ \cos y\,\frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y}.$$

We want to write this in terms of x.
Since $\sin y = x$, we have $\cos y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}$.
What shoule be $\cos y$ then ? $\sqrt{1-x^2}$ or $-\sqrt{1-x^2}$?
As you know, the range of $\sin^{-1}x$ is $\frac{-\pi}{2}$ to $\frac{\pi}{2}$, that is y lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. In this interval $\cos y$ is positive.
Therefore, we should take $\cos y = \sqrt{1-x^2}$.

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

$$\text{Thus}\ \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}.$$

Example 3.16

Find $\frac{d}{dx}(\cos^{-1}x)$.

Solution

First Method : Put $y = \cos^{-1}x$ and proceed as in Example 3.15.

We get the answer $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$.

Second Method : Using the formula, $\sin\left(\frac{\pi}{2} - x\right) = \cos x$, one proves easily that

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ is an identity.

Now, differentiating both sides with respect to x,

$$\frac{d}{dx}(\sin^{-1}x) + \frac{d}{dx}(\cos^{-1}x) = 0$$

$$\therefore \quad \frac{d}{dx}(\cos^{-1}x) = -\frac{d}{dx}(\sin^{-1}x) = \frac{-1}{\sqrt{1-x^2}}.$$

Example 3.17

Find the differential coefficient of $\tan^{-1}x$.

Solution

First Method : Let $y = \tan^{-1}x$.

Then $\tan y = x$.

Differentiating both sides with respect to x,

$$\sec^2 y \frac{dy}{dx} = 1.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

Thus $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$.

Second Method : We know that

$$\sin\theta = \tan\theta \,.\, \cos\theta = \frac{\tan\theta}{\sec\theta} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}.$$

Put $\tan\theta = x \therefore \theta = \tan^{-1}x$.

$$\text{Again,} \sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{x}{\sqrt{1+x^2}}.$$

$$\therefore \theta = \sin^{-1}\frac{x}{\sqrt{1+x^2}}$$

$$\boxed{\therefore \tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}}}$$

$$\boxed{\therefore \frac{d}{dx}(\tan^{-1}x) = \frac{d}{dx}\left(\sin^{-1}\frac{x}{\sqrt{1+x^2}}\right)}$$

$$= \frac{1}{\left(1-\frac{x^2}{1+x^2}\right)^{\frac{1}{2}}} \cdot \frac{\sqrt{1+x^2}\,.\,1 - x.\frac{1}{2\sqrt{1+x^2}}.\,2x}{(1+x^2)} \quad \text{(by chain rule)}$$

$$= \sqrt{1+x^2} \quad \frac{(1+x^2)-x^2}{\sqrt{1+x^2}\,.\,(1+x^2)}$$

$$= \frac{1}{1+x^2}\,.$$

$$\therefore \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}.$$

Example 3.18

Differentiate $\tan^{-1}\log x$.

Solution

$$\frac{d}{dx}(\tan^{-1}\log x) = \frac{1}{1+(\log x)^2} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{x[\,1+(\log x)^2\,]}$$

Example 3.19

Differentiate $\cos^{-1}(4x^3-3x)$.

Solution

$$\frac{d}{dx}[\cos^{-1}(4x^3-3x)\,] = \frac{-1}{\sqrt{1-(4x^3-3x)^2}}(12x^2-3)$$

$$= \frac{-3(4x^2-1)}{\sqrt{(4x^2-1)^2(1-x^2)}} \quad \text{(after simplification)}$$

$$= \frac{-3}{\sqrt{1-x^2}}\,.$$

Remark

Later you will learn a better method of doing this problem. See § 3.10, Example 3.27.

EXERCISE 3.6

1. Differentiate $\cot^{-1}x$ in two ways : (i) differentiating $\cot y = x$ (ii) using a relation between $\tan^{-1}x$ and $\cot^{-1}x$.

2. Prove that $\sec^{-1}x = \cos^{-1}\dfrac{1}{x}$.

   Use this to find $\dfrac{d}{dx}(\sec^{-1}x)$.

3. Prove that $\sec^{-1}x = \tan^{-1}\sqrt{x^2-1}$.

   Use this to find $\dfrac{d}{dx}(\sec^{-1}x)$.

4. Use $\dfrac{d}{d\theta}(\sec\theta) = \sec\theta.\tan\theta$ to find $\dfrac{d}{dx}(\sec^{-1}x)$

   Differentiate the following functions (5 to 10):

5. $\operatorname{cosec}^{-1}x$

6. $\sin^{-1}(\cos x)$

7. $\sin(\tan^{-1}x)$

8. $\sin^{-1}2x$

9. $e^{\sin^{-1}(x+1)}$

10. $(\cot^{-1}x)^2$

11. Find $\dfrac{d}{dx}(\cos^{-1}x)$ by differentiating $\cos y = x$.

12. Prove that $\dfrac{d}{dx}\left(\cos^{-1}\sqrt{\dfrac{1+x}{2}}\right) = \dfrac{-1}{2\sqrt{1-x^2}}$.

3.7 Differentiation of Implicit Functions

Hitherto, we have calculated $\dfrac{dy}{dx}$ when y is explicitly given as a function of x. However there are occasions when an equation involving x and y is given in such a way that y depends on x implicitly. In this section you learn to find $\dfrac{dy}{dx}$ in such cases. The following examples illustrate the method.

Example 3.20

Find $\dfrac{dy}{dx}$ where $2x^2 - 3xy + 4y^2 = 8$

Solution

Differentiating both sides of the equation with respect to x,

$$4x - 3\left(x\frac{dy}{dx} + y\right) + 8y\frac{dy}{dx} = 0.$$

Grouping the terms involving $\dfrac{dy}{dx}$ on one side and the others on the other side.

$$-3x\frac{dy}{dx} + 8y\frac{dy}{dx} = 3y - 4x$$

$$\therefore \quad \frac{dy}{dx}(8y - 3x) = 3y - 4x$$

$$\therefore \quad \frac{dy}{dx} = \frac{3y - 4x}{8y - 3x} = \frac{4x - 3y}{3x - 8y}.$$

In this differentiation above, we have used the chain rule while differentiating terms containing y. For example,

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \cdot \frac{d(y)}{dx} = 2y \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}(xy) = y\frac{d}{dx}(x) + x\frac{d}{dx}(y) = y.1 + x\frac{dy}{dx} = y + x\frac{dy}{dx}.$$

Remark

The answer gives $\dfrac{dy}{dx}$ as a function of both x and y. We eliminate y in the answer only when it is possible.

Example 3.21

If $\sin(x+y) = \dfrac{1}{2}$, find $\dfrac{dy}{dx}$.

Solution

Differentiating both sides with respect to x, we get $\cos(x+y)\left(1 + \dfrac{dy}{dx}\right) = 0$.

But $\cos(x+y)$ cannot be 0, since $\sin(x+y) = \frac{1}{2}$.

$$\therefore 1+\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = -1.$$

Remark

Sometimes, instead of differentiating as it is, it is better to rewrite the equation in a simpler form before differentiating. This is seen in the next example.

Example 3.22

If $y = b\tan^{-1}\left(\frac{x}{a} + \tan^{-1}\frac{y}{x}\right)$, find $\frac{dy}{dx}$.

Solution

First Method : Differentiating both sides with respect to x.

$$\frac{dy}{dx} = \frac{b\left[\frac{1}{a} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\,\frac{x\frac{dy}{dx}-y}{x^2}\right]}{1+\left(\frac{x}{a}+\tan^{-1}\frac{y}{x}\right)^2}$$

$$= \frac{b\left[\frac{1}{a} + \frac{x\frac{dy}{dx}-y}{x^2+y^2}\right]}{1+(\frac{x}{a}+\tan^{-1}\frac{y}{x})^2}$$

$$\therefore \frac{dy}{dx}\left[1+(\frac{x}{a}+\tan^{-1}\frac{y}{x})^2 - \frac{bx}{x^2+y^2}\right] = \frac{b}{a} - \frac{by}{x^2+y^2}.$$

$$\therefore \frac{dy}{dx} = \frac{\frac{b}{a} - \frac{by}{x^2+y^2}}{1+(\frac{x}{a}+\tan^{-1}\frac{y}{x})^2 - \frac{bx}{x^2+y^2})}$$

Better Method : Rewrite the equation as

$$\tan\frac{y}{b} = \frac{x}{a} + \tan^{-1}\frac{y}{x}.$$

Differentiating both sides with respect to x,

$$\frac{1}{b}\sec^2\frac{y}{b}\cdot\frac{dy}{dx} = \frac{1}{a} + \frac{x\frac{dy}{dx} - y}{\left(1+\frac{y^2}{x^2}\right).x^2}$$

$$= \frac{1}{a} + \frac{x\frac{dy}{dx} - y}{x^2+y^2}$$

$$\therefore \frac{dy}{dx}\left(\frac{1}{b}\sec^2\frac{y}{b} - \frac{x}{x^2+y^2}\right) = \frac{1}{a} - \frac{y}{x^2+y^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{a} - \frac{y}{x^2+y^2}}{\frac{1}{b}\sec^2\frac{y}{b} - \frac{x}{x^2+y^2}}.$$

EXERCISE 3.7

Find $\frac{dy}{dx}$ when x and y are connected by the following relations :

1. $x^2+y^2=r^2$
2. $xy=c^2$
3. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
4. $y^2=4ax$
5. $ax^2+2hxy+by^2+2gx+2fy+c=0$
6. $(x^2+y^2)^2=xy$
7. $x^3+y^3=3axy$
8. $xy^2-x^2y=4$

3.8 Logarithmic Differentiation

In some problems, it is easier to find $\frac{dy}{dx}$, by first taking logarithms and then differentiating. We meet such problems in this section. Such process is called logarithmic differentiation. This is usually done in two kinds of problems : First when the function is a product of many simpler functions. In this case logarithm converts the product into a sum and facilitates differentiation. Secondly, when the variable x occurs in the exponent. In this case logarithm brings it to a more manageable form. We work out one example for each of these two instances.

Example 3 23

Differentiate $\sin x.\ \sin 2x\ \sin 3x.\ \sin 4x$

Solution

Let $y = \sin x\ \sin 2x\ \sin 3x.\ \sin 4x.$

Then $\log y = \log \sin x + \log \sin 2x + \log \sin 3x + \log \sin 4x.$

Differentiating both sides, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{\sin x}.\cos x + \frac{1}{\sin 2x}.\cos 2x.2 + \frac{1}{\sin 3x}\ \cos 3x.3 + \frac{1}{\sin 4x}.\cos 4x\ 4$$

$$= \cot x + 2\cot 2x + 3\cot 3x + 4\cot 4x$$

$$\therefore \frac{dy}{dx} = \sin x\ \sin 2x.\ \sin 3x\ \sin 4x\,[\cot x + 2\cot 2x + 3\cot 3x + 4\cot 4x\,].$$

Remark

If we do this problem without taking logarithms, it will be cumbersome, because the product rule is to be applied many times.

Example 3.24

Differentiate x^x

Remark

If a and b are two real numbers, then a^b is defined as $e^{b \log a}$ whenever a is positive.

Solution

Let $y = x^x$. Then $\log y = x \log x.$

Differentiating both sides with respect to x, we have

$$\frac{1}{y}\ \frac{dy}{dx} = x.\frac{1}{x} + \log x.1$$

$$= 1 + \log x$$

$$\therefore \frac{dy}{dx} = y\,(1 + \log x)$$

$$= x^x(1 + \log x).$$

EXERCISE 3.8

Differentiate the following functions :

1. $(\sin x)^x$
2. $x^2 e^x \sin x$
3. 2^x
4. $e^x \cos^3 x \sin^2 x$
5. $(x + 1)^2 (x + 2)^3 (x + 3)^4$
6. $e^{(x+1)^2}$
7. $x^{\sin x + \cos x}$
8. $(2x + 3)^{x-5}$
9. $\dfrac{8^x}{x^8}$
10. $\sqrt{(x-1)(x-2)(x-3)(x-4)}$

3.9 Differentiation of Parametric Forms

Sometimes x and y are given as functions of another variable t. We call t, the parameter in which x and y are expressed. We find $\dfrac{dy}{dx}$ in such cases as explained below.

Let $x = f(t)$ and $y = g(t)$.

Let Δx and Δy be increments in x and y respectively corresponding to the increment Δt in t.

$\therefore\ y + \Delta y = g(t + \Delta t)$ and $x + \Delta x = f(t + \Delta t)$.

We get

$$\Delta y = g(t + \Delta t) - g(t)$$
$$\Delta x = f(t + \Delta t) - f(t)$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{g(t+\Delta t)-g(t)}{f(t+\Delta t)-f(t)}$$

$$= \frac{g(t+\Delta t)-g(t)}{\Delta t} \div \frac{f(t+\Delta t)-f(t)}{\Delta t}.$$

As $\Delta t \to 0$, $\Delta x \to 0$ and $\Delta y \to 0$.

Taking limits, we get

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta t \to 0} \frac{g(t+\Delta t)-g(t)}{\Delta t} \div \lim_{\Delta t \to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t}$$

$$= \frac{dy}{dt} \div \frac{dx}{dt}.$$

Thus, when x and y are functions of the parameter t,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}.$$

Example 3.25

If $x = a \cos \theta$ and $y = b \sin \theta$, find $\frac{dy}{dx}$.

Solution

$$\frac{dx}{d\theta} = -a \sin \theta$$

$$\frac{dy}{d\theta} = b \cos \theta$$

$$\therefore \frac{dy}{dx} = \frac{b \cos \theta}{-a \sin \theta} = -\frac{b}{a} \cot \theta.$$

Example 3.26

Find $\frac{dy}{dx}$ when $x = \frac{3at}{1+t^2}$ and $y = \frac{3at^2}{1+t^2}$.

Solution

$$\frac{dx}{dt} = \frac{(1+t^2)3a - 3at.2t}{(1+t^2)^2} = \frac{1-3at^2}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{(1+t^2)6at - 3at^2.2t}{(1+t^2)^2} = \frac{1}{(1+t^2)^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{1-3at^2}$$

EXERCISE 3.9

Find $\frac{dy}{dx}$ when

1. $x=at^2,\ y=2at$
2. $x=ct,\ y=\frac{c}{t}$
3. $x=a\sec\theta,\ y=b\tan\theta$
4. $x=\sin 2t,\ y=2\cos t$
5. $x=\log t,\ y=\sin t$
6. $x=a(\theta-\sin\theta),\ y=a(1-\cos\theta)$
7. $x=2\cos^2\theta,\ y=2\sin^2\theta$
8. $x=\cos\theta+\cos 2\theta,\ y=\sin\theta+\sin 2\theta$

3.10 Differentiation by Substitution

Sometimes, it is easier to differentiate, by making substitutions. We see such examples in this section. Usually these examples involve inverse trigonometric functions.

Example 3.27

Differentiate $\cos^{-1}(4x^3-3x)$.

Solution

First Method : If we differentiate directly,

$$\frac{d}{dx}\left[\cos^{-1}(4x^3-3x)\right] = -\frac{1}{\sqrt{1-(4x^3-3x)^2}}\cdot\frac{d}{dx}(4x^3-3x)$$

$$= \frac{3-12x^2}{\sqrt{1-(4x^3-3x)^2}}.$$

Second Method : Substitute $x=\cos\theta$.

Then $4x^3-3x=4\cos^3\theta-3\cos\theta$

$=\cos 3\theta$ [by a known formula]

$\therefore \cos^{-1}(4x^3-3x)=3\theta$

$$\therefore \frac{d}{dx}[\cos^{-1}(4x^3-3x)] = \frac{d}{dx}(3\theta)$$

$$= \frac{d}{dx}(3\theta)=3\frac{d\theta}{dx}$$

$$= 3.\frac{d}{dx}(\cos^{-1}x)$$

$$= \frac{-3}{\sqrt{1-x^2}}.$$

Remark

The two answers are apparently different, but actually same. One can prove that

$$\frac{3-12x^2}{\sqrt{1-(4x^3-3x)^2}} = \frac{-3}{\sqrt{1-x^2}}.$$

Example 3.28 .

Differentiate $\tan^{-1}(\sqrt{1+x^2}-x)$.

Solution

Put $x=\tan\theta$.

Then $1+x^2=\sec^2\theta$

$\sqrt{1+x^2}-x=\sec\theta-\tan\theta$

$$=\frac{1-\sin\theta}{\cos\theta}$$

$$=\frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}$$

$$=\frac{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)}$$

$$= \frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}$$

$$= \frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}$$

$$= \frac{\tan\frac{\pi}{4} - \tan\frac{\theta}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{\theta}{2}}$$

$$= \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right).$$

$$\therefore \tan^{-1}\left(\sqrt{1-x^2} - x\right) = \frac{\pi}{4} - \frac{\theta}{2}.$$

$$\therefore \frac{d}{dx}\left(\tan^{-1}\left(\sqrt{1+x^2}-x\right)\right) = \frac{d}{dx}\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$= \frac{d}{d\theta}\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \cdot \frac{d\theta}{dx}$$

$$= \frac{-1}{2} \cdot \frac{d\theta}{dx}$$

$$= \frac{-1}{2}\frac{d}{dx}(\tan^{-1}x)$$

$$= \frac{-1}{2(1+x^2)}.$$

EXERCISE 3.10

Differentiate the following functions, using suitable substitutions :

1. $\tan^{-1}\dfrac{2x}{1-x^2}$

2. $\sin^{-1}(3x-4x^3)$
3. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$
4. $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$
5. $\sec^{-1}\dfrac{1}{1-2x^2}$
6. $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$
7. $\sin^{-1}(2x\sqrt{1-x^2})$
8. $\cos^{-1}\sqrt{\dfrac{1+x}{2}}$

3.11 Second Derivatives

For some physical and geometrical concepts, we need the notion of the second derivative. This is developed in this section.

We have seen that if f is a differentiable function, then f' is another function. If this f' is also differentiable, then its derivative is denoted by f'', and is called the second derivative of f.

If $y=f(x)$, then we have seen that $f'(x)$ is also denoted by $\dfrac{dy}{dx}$ or y'. Similarly f'' is denoted by $\dfrac{d^2y}{dx^2}$ or y''

Example 3.29

Find the second derivative of $e^x+\sin x$.

Solution

Let $y=e^x+\sin x$.

Differentiating, $\dfrac{dy}{dx}=e^x+\cos x$

Differentiating once again,

$$\frac{d^2y}{dx^2}=e^x-\sin x.$$

Example 3.30

If $y = \log x - x$, find $\dfrac{d^2y}{dx^2}$.

Solution

$y = \log x - x$

Differentiating this,

$$\frac{dy}{dx} = \frac{1}{x} - 1.$$

Differentiating once again,

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}.$$

EXERCISE 3.11

Find the second derivatives of the following functions :

1. x^2
2. $ax^3 + bx^2 + cx + d$
3. x^n
4. $x \sin x$
5. e^{kx}
6. $\log \log x$
7. x
8. If $y = 2\sin x + 3\cos x$, prove that $y + \dfrac{d^2y}{dx^2} = 0$.
9. If $y = \tan x$, prove that $\dfrac{d^2y}{dx^2} = 2y\dfrac{dy}{dx}$.
10. If $y = \sin^{-1}x$, prove that $\dfrac{d^2y}{dx^2} = \dfrac{x}{(1-x^2)^{\frac{3}{2}}}$

3.12 The n^{th} Derivative

For each positive integer n, one can define the n^{th} derivative of a function. (Sometimes it may not exist). When $n = 1$, the n^{th} derivative of f is the usual derivative f'. When $n = 2$, the n^{th} derivative is the second derivative seen in the last section For $n = 3$, it is the derivative of the second derivative; and so on. After finding the first and second derivatives, we can in many cases guess the formula for the n^{th} derivative The rigorous proof needs induction, and we do not require this much of rigour at present.

Notation

The n^{th} derivative of a function $f(x)$ will be denoted by $f^{(n)}$.

When it is written as $y = f(x)$, the nth derivative is denoted by either $y^{(n)}$ or y_n or $\dfrac{d^ny}{dx^n}$.

Example 3.31

Find the n^{th} derivative of $\sin x$

Solution

Let $y = \sin x$.

Then $\dfrac{dy}{dx} = \cos x = \sin\left(x + \dfrac{\pi}{2}\right)$.

Differentiating again

$$\frac{d^2y}{dx^2} = -\sin x = \sin(x + \pi).$$

One can prove that

$$\frac{d^ny}{dx^n} = \sin\left(x + n\frac{\pi}{2}\right) \text{ for all } n = 1, 2, 3, \ldots$$

Example 3.32

Find the n^{th} derivative of $\dfrac{1}{x}$.

Solution

Let $y = \dfrac{1}{x}$.

Then $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$

$$\frac{d^2y}{dx^2} = -\left(\frac{-2}{x^3}\right) = \frac{2}{x^3}$$

$$\frac{d^3y}{dx^3} = -\frac{2\,3}{x^4} = -\frac{6}{x^4}.$$

Thus proceeding, one proves

$$\frac{d^ny}{dx^n} = (-1)^n \, . \, \frac{n!}{x^{n+1}}.$$

Example 3 33

Find the n^{th} derivative of x^n.

Solution

The first derivative of x^n is $n\,x^{n-1}$.

The second derivative of x^n

$= $ the derivative of $n\,x^{n-1}$

$= n.\,(n-1)\,x^{n-2}$.

The third derivative of x^n

$= $ the derivative of $n\,(n-1)\,x^{n-2}$

$= n\,(n-1)\,(n-2)\,x^{n-3}$

Thus proceeding,

the n^{th} derivative of x^n

$= n(n-1)\,(n-2)\ldots\big(n-(n-1)\big).\,x^{n-n}$

$= n!\,x^0$

$= n!$.

EXERCISE 3.12

Find the n^{th} derivatives of the following functions :

1. $\cos x$
2. $\sin 2x$
3. $\cos a^x$
4. e^{ax}
5. x^{n+1}

6. $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$

7. $(ax+b)^n$

8. x^m (*Hint* : The cases $m<n$, $n<m$, $n = m$ are to be treated separately)

A List of Formulas to Remember

Sl no.	*The function y*	*The derivative* $\frac{dy}{dx}$
1.	Constant Function	zero
2.	x^n	$n\,x^{n-1}$
3	$\sin x$	$\cos x$
4.	$\cos x$	$-\sin x$
5.	$\tan x$	$\sec^2 x$
6.	$\cot x$	$-\operatorname{cosec}^2 x$
7	$\sec x$	$\sec x.\tan x$
8.	$\operatorname{cosec} x$	$-\operatorname{cosec} x.\cot x$
9.	e^x	e^x
10.	$\log x$	$\frac{1}{x}$
11.	$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
12.	$\cos^{-1}x$	$\frac{-1}{\sqrt{1-x^2}}$
13.	$\tan^{-1}x$	$\frac{1}{1+x^2}$
14.	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$

Product Rule : $(fg)' = fg' + f'g$

Quotient Rule $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$;

Chain Rule : $(f \circ g)'(x) = f'(g(x)).\, g'(x)$.

MISCELLANEOUS EXERCISE ON CHAPTER 3

1. Differentiate $x \sin x$ from the first principle.

2. If $y=x^x$, find $\dfrac{d^2y}{dx^2}$.

3. If $x=c \tan \theta$ and $y=c \cot \theta$, find $\dfrac{dy}{dx}$.

4. Differentiate x^{x^x}

5. If $xy^3-yx^3=x$, find $\dfrac{dy}{dx}$

6. If the derivative of $\tan^{-1}(a+bx)$ takes the value 1 at $x=0$, prove that $a=b^2$.

7. Differentiate $\dfrac{x+2}{x^2-3}$ and find the value of the derviative at $x=0$.

Differentiate the following functions (9 to 28) :

9. $\log_x x$

10. $x \tan^{-1}x$

11. $\sec(3x+4)$

12. $x \log x . \log \log x$

13. $\left(\dfrac{1}{x}\right)^x$

14. $\dfrac{\sin 2x}{e^x}$

15. $\operatorname{cosec}^{-1}\dfrac{1+x^2}{2x}$

16. $\cos(\log x)$

17. $\log \dfrac{1+x}{1-x}$

18. $\sin^m x \cos^n x$

19. $\log\left(x+\sqrt{x^2+a^2}\right)$

20. $x(x-2)\sqrt{x-3}$

21. $\sqrt{1+\tan x}$

22. $\sin(e^x \log x)$

CHAPTER 4

APPLICATIONS OF DERIVATIVES

Derivatives have a wide range of applications in science and engineering as well as in social sciences. In this chapter we shall consider a few applications of the derivative

4.1 Motion in a Straight Line

Suppose a particle P is moving in a straight line. We take a point O on the straight line as origin and set up a coordinate system on the line, that is, we associate a real number with each point on the line. The directed distance of the particle from the origin is a function of time. Let the particle P be at the point s at time t. Then $s=f(t)$.

As you already know, the velocity of the particle is the rate of change of its distance (from the origin). Let the position of the particle be the point $(s+\Delta s)$ at time $t+\Delta t$, so that $s+\Delta s=f(t+\Delta t)$. Then average velocity of the particle between the points

$$s \text{ and } s+\Delta s=\frac{\Delta s}{\Delta t}.$$

The velocity v (instantaneous velocity) of the particle at the point s (at time t) is the limiting value of $\frac{\Delta s}{\Delta t}$ as $\Delta t \to 0$.

$$\therefore\ v=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}=\lim_{\Delta t\to 0}\frac{(s+\Delta s)-s}{\Delta t}=\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}=f'(t)$$

Similarly, if v is the velocity of the particle at time t and $v+\Delta v$ is the velocity at $t+\Delta t$, then average acceleration of the particle between the points s and $s+\Delta s$

$$=\frac{\Delta v}{t}=\frac{(v+\Delta v)-v}{\Delta t}=\frac{f'(t+\Delta t)-f'(t)}{t}.$$

The acceleration a at the point s is the limiting value of the average acceleration of the particle between the points s and $s\ \Delta s+$

$$=\lim_{\Delta t\to 0}\frac{f'(t+\Delta t)-f'(t)}{\Delta t}=f'(t).$$

Thus when the distance travelled by a particle (from an origin) is given as a function of time, say $s=f(t)$, then

$f'(t)\ \left(=\frac{ds}{dt}\right)$ represents its velocity and $f''(t)\ \left(=\frac{d^2s}{dt^2}\right)$ represents its acceleration at time t.

Example 4.1

A particle is moving along a straight line according to the formula $s = 12t - 3t^2$, where s is in metres and t is in seconds. Find its velocity and acceleration.

Solution

We have

$s = 12t - 3t^2$. Differentiating with respect to t,

$$\frac{ds}{dt} = 12 - 6t.$$

This gives the velocity at time t.

Differentiating once again with respect to t,

$$\frac{d^2s}{dt^2} = -6.$$

This gives the acceleration at time t.

We note that the acceleration is the same at all times t.

The negativeness of the acceleration means that the velocity is decreasing. Sometimes it (negative acceleration) is called retradation.

Example 4.2

The distance s in metres described by a particle in t seconds is given by $s = ae^t + \dfrac{b}{e^t}$.

Show that the acceleration of the particle at time t is equal to the distance travelled by it upto time t

Solution

We have

$$s = ae^t + \frac{b}{e^t}$$

Differentiating with respect to t,

$$\frac{ds}{dt} = ae^t - \frac{b}{e^t}.$$

Differentiating once again,

$$\frac{d^2s}{dt^2} = ae^t + \frac{b}{e^t}.$$

We note that

$$\frac{d^2s}{dt^2} = s.$$

In other words,

acceleration at time t = distance travelled upto time t.

Example 4.3

A particle is moving in a straight line according to the formula $s = t^3 - 9t^2 + 3t + 1$, where s is measured in metres and t in seconds. When the velocity is -24 m/s, find the acceleration.

Solution

We have

$s = t^3 - 9t^2 + 3t + 1.$

The velocity is given by $\dfrac{ds}{dt} = 3t^2 - 18t + 3 = 3(t^2 - 6t + 1).$

If this is equal to -24, then

$3(t^2 - 6t + 1) + 24 = 0$

That is, $t^2 - 6t + 9 = 0$

$\therefore\ t = \pm\, 3$

The acceleration is given by

$$\frac{d^2s}{dt^2} = 3(2t - 6) = 6(t - 3)$$

$$\left(\frac{d^2s}{dt^2}\right)_{t\,=\,3} = 0 \text{ and } \left(\frac{d^2s}{dt^2}\right)_{t\,=\,-3} = -36\,.$$

Thus, when the velocity is -24 m/s, the acceleration is either 0 m/s^2 or -36 m/s^2.

To be clear, the velocity is -24 m/s at two instants, namely, when $t = 3$ seconds and when $t = -3$ seconds or when $s = -44$ m or -116 m. At the point $s = -44$ m the acceleration is 0 m/s^2 and at the point $s = -116$ m, it is -36 m/s^2.

EXERCISE 4.1

1. In the following (i) – (iv), find the velocity of a particle moving on a line at $t = 2.25$ seconds, if its position s is in metres and time t is in seconds :

   (i) $s = \sin \dfrac{t}{9}$

   (ii) $s = t^2$

   (iii) $s = t - 1$

   (iv) $s = \log t$

2. Find the acceleration of the particle at $t = 3$ in the Problems (i) to (iv) above.

3. A particle is moving on a line, where its position s in metres is a function of time t in seconds given by $s = t^3 + at^2 + bt + c$, where a, b, c are constants. It is known that at

$t=1$, second the position of the particle is given by $s=7$ metres, velocity is 7 m/s and acceleration is 12 m/s^2. Find the values of a, b and c.

4. A car is running on a straight road The distance travelled and time taken are connected by the formula $s=t^2-2t$, where t is measured in hours and s in kilometres. When the odometer reading is 15, what is the speedometer reading ? (Note : odometer measures s, and the speedometer measures velocity. You may take $t\geq 0$ here).

5. A particle moving according to the formula $s=10+20t-t^2$, starts from a distance of 10 metres from a mark, and moves in a line farther and farther from the mark. How far from the mark does it go, before it starts moving in the opposite direction ?

4.2 Motion Under Gravity

One particular instance of motion in a straight line is the motion of a falling body under gravity. The acceleration of the falling body due to gravity has been calculated as $g=32$ feet/second^2 or 9.8 metres/second^2, towards the centre of the earth. In this section, we use differentiation to some practical problems concerning this motion Theoretically, however, there is nothing new in this section.

Example 4.4

A ball thrown vertically upwards, moves according to the formula $s=13.8t-4.9t^2$, where s is in metres and t is in seconds. Find the following :

(a) Its acceleration at $t=1$

(b) Its velocity at $t=1$

(c) The maximum height reached by the ball.

Solution

We have

$s=13.8t-4.9t^2$.

Differentiating,

$$\frac{ds}{dt}=13.8-9.8t.$$

Differentiating once again, we get

$$\frac{d^2s}{dt^2}=-9.8.$$

$\therefore$ Acceleration at $t=1$ is $\left(\dfrac{d^2s}{dt^2}\right)_{t=1}=-9.8$ m/s^2.

(In fact, it is the same for all values of t).

The velocity at $t=1$ is $\left(\dfrac{ds}{dt}\right)_{t=1}=(13.8-9.8\times 1)=4.0$ m/s.

We note that the velocity becomes zero when

$$t = \frac{13.8}{9.8}$$

and after that, it becomes negative, indicating that the ball comes downwards.

Therefore, the maximum height is reached when $t = \frac{13.8}{9.8}$ For this value of t,

$$\begin{aligned}
s &= 13.8\left(\frac{13.8}{9.8}\right) - (4.9)\left(\frac{13.8}{9.8}\right)^2 \\
&= \frac{1}{9.8}\left[(13.8)^2 - \frac{1}{2}(13.8)^2\right] \\
&= \frac{(13.8)^2}{2\times(9.8)}\left[2-1\right] = \frac{(13.8)^2}{2\times(9.8)} \\
&= 9.716 \text{ approximately.}
\end{aligned}$$

The maximum height reached by the ball is 9.716 metres approximately

Example 4.5

The motion of a stone thrown vertically upwards satisfies an equation of the form $s = at^2 + bt$ when s and t are measured in metres and seconds respectively. If the maximum height reached by the stone is 4.9 metres and if its acceleration is – 9.8 m/s², find its height after half a second.

Solution

We have

$s = at^2 + bt.$

Differentiating,

$$\frac{ds}{dt} = 2at + b.$$

Differentiating once again,

$$\frac{d^2s}{dt^2} = 2a.$$

It is given that the acceleration is – 9.8 m/s².

$$\therefore\ 2a = -9.8$$

$$\text{or}\quad a = -4.9$$

$$\therefore\ \frac{ds}{dt} = b - 9.8t.$$

$\frac{ds}{dt}$ becomes zero when $t=\frac{b}{9.8}$ and negative for greater values of t. (If b were negative, there would be no upward motion at all). The maximum height reached is given by

$s=at^2+bt$ with $t=\frac{b}{9.8}$.

This is, when $s=-4.9\frac{b^2}{(9.8)^2}+\frac{b^2}{9.8}=\frac{b^2}{9.8}(1-\frac{1}{2})=\frac{b^2}{19.6}$.

From the given data $\frac{b^2}{19.6}=4.9$

$\therefore\ b^2=4.9\times 19.6$.

$\therefore\ b=9.8$ (as already mentioned, b cannot be negative)

$\therefore$ The equation of motion of the stone becomes

$s=-4.9t^2+9.8t$.

When $t=\frac{1}{2}$, $s=-\frac{4.9}{4}+\frac{9.8}{2}=-\frac{4.9}{4}+4.9$

$=4.9\,(1-\frac{1}{4})=\frac{14.7}{4}=3.675$.

The stone is at a height of 3.675 m after half a second.

EXERCISE 4.2

1. A stone, thrown upwards, has its equation of motion $s=490t-4.9t^2$. What is the maximum height reached by it ?
2. If a ball, thrown vertically upwards, has equation of motion $s=ut+\frac{1}{2}at^2$ in metres and seconds and if $a=-9.8$ m/s², find the maximum height reached in terms of u
3. The maximum height is reached in 3 seconds by a stone thrown up vertically and moving under the equation $s=ut-4.9t^2$, where s is in metres and t is in seconds. Find the value of u.
4. A ball thrown vertically upwards, falls back on the ground after 8 seconds. Assuming that the equation of motion is of the form $s=ut-4.9t^2$, where s is in metres and t is in seconds, find the velocity at $t=0$. Find also the velocity at $t=1$.

5. Someone standing on a pole of height 9.8m metres throws a stone vertically upwards. It moves in a vertical line slightly away from the line of the pole, and falls on the ground. If its equation of motion, in metres and seconds, is $s = 19.6t - 4.9t^2$, how much time does it take for the upward motion.

6. A particle is moving in a vertical line as per the equation $s = 100t - 4.9t^2$, where s is in metres and t is in seconds. What is its velocity at $t = 1$? At what time is its velocity zero ? What is its acceleration at $t = 1$? What is the maximum value of s ?

7. When a stone is thrown upwards, on certain planet, its equation of motion is $s=10t-3t^2$ in metres and seconds. After how many seconds will it fall back (on the planet) again ? What is the acceleration (due to gravity of that planet) ?

8. Two stones are thrown up simultaneously. Their equations of motions are respectively $s = 19.6t - 4.9t^2$ for the first stone and $s = 9.8t - 4.9t^2$ for the second stone What is the height of the second stone, when the height of the first stone is maximum ?

4.3 Rate of Change of Quantities

We have already seen that the derivative $\frac{ds}{dt}$ represents velocity, the rate of change of distance with respect to time t. In a similar fashion, whenever one quantity y varies with another quantity x, according to the rule $y=f(x)$, then $f'(x_0)$ represents the rate of change of y with respect to x at $x=x_0$ In this section, we see some examples of this sort.

Further we have, if both x and y are varying with t (i.e. x, y are functions of t), then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \qquad \text{(by chain rule)}$$

$$= f'(x) \, . \, \frac{dx}{dt} \, .$$

Thus, the rate of change of one variable can be calculated if the rate of change of the other variable is known.

Example 4.6

A stone is dropped into a quiet lake and waves move in circles at a speed of 4 cm. per second. At the instant when the radius of the circular wave is 10 cm how fast is the enclosed area increasing ?

Solution

If r is the radius of the circular wave and if A is the enclosed area at time t, then we know that

$$A = \pi r^2.$$

Differentiating this with respect to time t, we have

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

It is given that $\frac{dr}{dt} = 4$ cm/s.

$\therefore$ when $r = 10$ cm,

$$\frac{dA}{dt} = 2\pi.20.4 \text{ cm}^2/\text{s}$$
$$= 80\pi \text{ cm}^2/\text{s}.$$

The enclosed area is increasing at the rate of 80 cm²/s when $r = 10$ cm.

Example 4.7

A man 2 metres high walks at a uniform speed of 5 metres/hour away from a lamp-post 6 metres high. Find the rate at which the length of his shadow increases.

Solution

In Fig. 4.1, let AB be the lamp-post, the lamp being at the position B, and let MN be the man at a particular time t and let $AM = l$m.

Then MS is the shadow of the man. First we want to express the length MS in terms of the length AM.

Since the triangles ASB and MSN are similar,

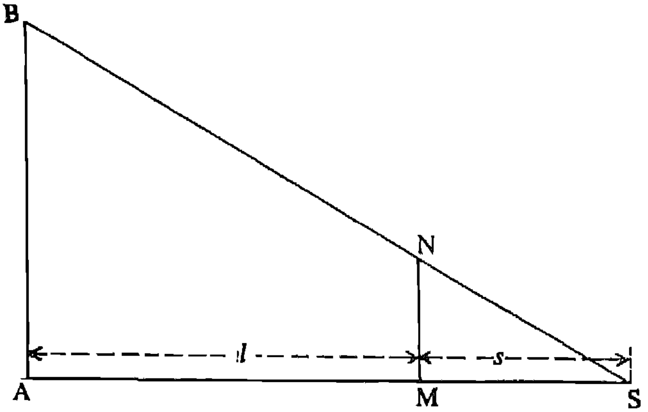


Fig. 4.1

we have $\dfrac{MS}{AS} = \dfrac{MN}{AB}$.

But $MN = 2$ metres and $AB = 6$ metres.

Let $AS = s\ m$.

Thus $\dfrac{MS}{AS} = \dfrac{2}{6} = \dfrac{1}{3}$ $\quad\therefore\ AS = 3s$.

It follows that $AM = 2\ s$.

In other words, $l = 2\ s$.

Therefore, $\dfrac{dl}{dt} = 2\dfrac{ds}{dt}$.

But it is given that $\dfrac{dl}{dt} = 5$ km/h.

It follows that $\dfrac{ds}{dt} = \dfrac{5}{2}$.

The length of the shadow increases at the rate of $\dfrac{5}{2}$ km/h.

EXERCISE 4.3

1. A man 160 cm tall, walks away from a source of light situated at the top of a pole 6 m high, at the rate of 1.1 m/s. How fast is the length of his shadow increasing when he is 1 m away from the pole?
2. An airforce plane is ascending vertically at the rate of 100 km/h. If the radius of the earth is r km, how fast is the area of the earth, visible from the plane, increasing at 3 minutes after it started ascending ? (visible area $A = \dfrac{2\pi r^2 h}{r+h}$, where h is the height of the plane above the earth.)
3. A balloon which always remains spherical, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 cm.
4. An edge of a variable cube is increasing at the rate of 3 cm per second. How fast is the volume of the cube increasing when the edge is 10 cm long ?

5. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-cordinate.

6. A ladder 5m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?

7. Water is dripping out from a conical funnel, at the uniform rate of 2 cm^2/s through a tiny hole at the vertex at the bottom When the slant height of the water is 4 *cm*, find the rate of decrease of the slant height of the water

8. Sand is pouring from a pipe at the rate of 12 cm^2/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand-cone increasing when the heigh is 4 cm ?

4.4 Increasing and Decreasing Functions

In this section, we use differentiation to decide whether a function is increasing or decreasing or neither.

To explain the notion of an increasing function, we start with an example The rate of US dollar in rupees, has been varying with time. In the period of two months, October and November 1988, the rate on some dates is tabulated below ·

Date	*Rupees per dollar*
4.10.88	14.5882
11.10.88	14.6113
14.10.88	14.6649
18.10.88	14.7341
21.10.88	14.7623
28.10.88	14.8456
4.11.88	14.9220
11.11.88	14.8887
18.11.88	15.0083

Here the rate is a function of time. We may take the rate to be defined at each point of time.

We observe that the rate has been increasing in October. But it is not so in November. In the month of November, after taking the value 14.9220 at one time, it takes the smaller value 14.8887 at a later time though once again it takes a higher value subsequently. That is why we say that the rate is not increasing in November. This situation leads us to the following definitions of increasing and decreasing functions

Definition

Let I be an open interval contained in the domain of a real function f. f is called an increasing function on I if whenever $x_1 < x_3$ in I, it is true that $f(x_1) < f(x_2)$. We may write this definition more briefly in symbolic form as follows .

$f(x)$ is an increasing function on I if $x_1 < x_2$ in $I \Rightarrow f(x_1) < f(x_2)$.

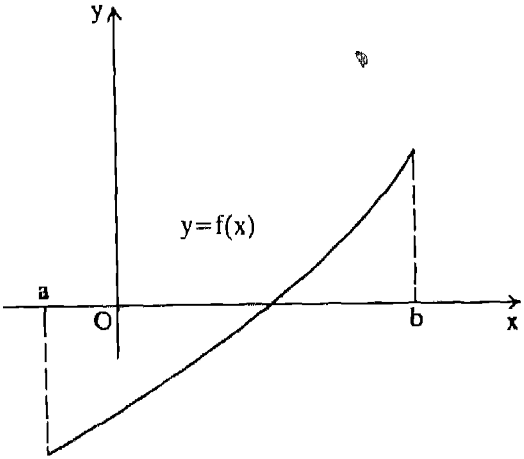


Fig. 4.2

The graph in Fig.4.2 is the graph of an increasing function $f(x)$ in (a, b).

Definition

A function f is a decreasing function on I if whenever $x_1 < x_2$ in I, it is true that $f(x_1) > f(x_2)$.

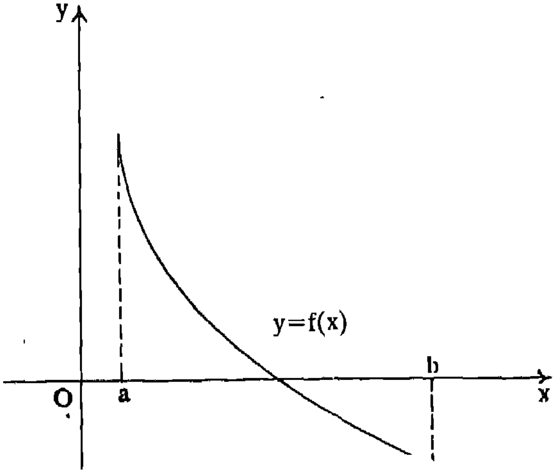


Fig. 4.3

In symbolic form a function f is decreasing function on I if $x_1 < x_2$ in $I \Rightarrow f(x_1) > f(x_2)$.
The graph in Fig. 4.3 is the graph of a decreasing function $f(x)$ in (a, b).

Remark

1. It is possible that a function f is neither increasing nor decreasing on a given I. The function $f(x)$ in Fig. 4.4 is neither increasing nor decreasing in (a, b). However, it is increasing in the intervals (a, c), (d, e) and (f, b) and decreasing in the intervals (c, d) and (e, f). In the example given in the beginning of this section, if I is the interval including the two months of October and November, we say the rate is not increasing on I.

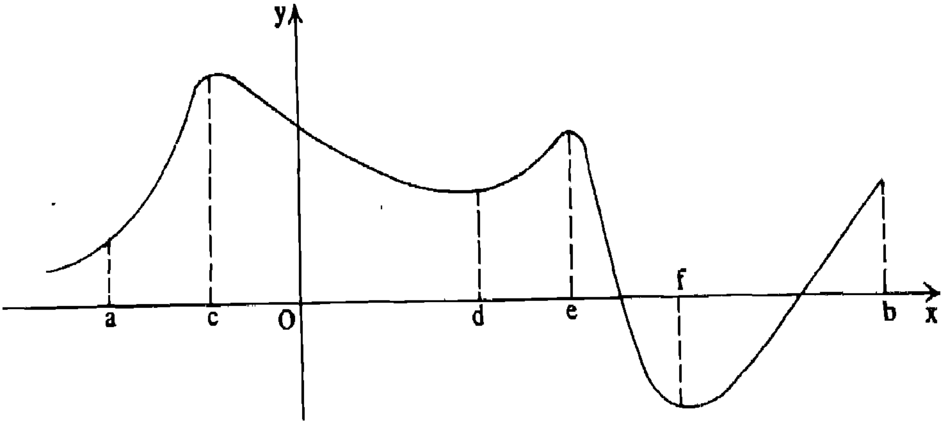


Fig. 4.4

2. In the above definition the interval I can be replaced by the real line R also.
3. What we have defined here are called "strictly increasing" functions by some authors. According to them a functions is increasing if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$. There could be functions which are increasing in this sense, but not according to our definition. For example, consider the function whose graph is shown in Fig. 4.5, which is increasing in $(0, c)$ according to them. According to our definition it is increasing only in the intervals $(0, a)$ and (b, c).

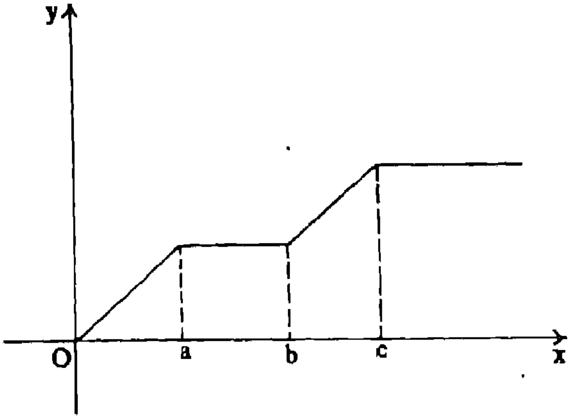


Fig. 4.5

Definition

A function f is said to be increasing at a point x_0, if there is an interval $I = (x_0 - h, x_0 + h)$ around x_0 such that for $x_1, x_2 \in I$,

$$x_0 < x_2 \Rightarrow f(x_0) < f(x_2)$$

and $x_1 < x_0 \Rightarrow f(x_1) < f(x_0)$.

Example 4.8

Show that the function $f(x) = 2x + 5$ is an increasing function on R.

Solution

Let $x_1, x_2 \in$ R and let $x_1 < x_2$.

$$x_1 < x_2$$

$$\therefore\ 2x_1 < 2x_2$$

$$\therefore\ 2x_1 + 5 < 2x_2 + 5$$

or $f(x_1) < f(x_2)$

$$\therefore\ x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

$\therefore$ f is an increasing function in R.

Example 4.9

Show that the function $f(x) = x^2$ is an increasing function in $(0, \infty)$.

Solution

Let $x_1, x_2 \in (0, \infty)$ and let $x_1 < x_2$.

Now, $$x_1 < x_2 \qquad \ldots (4.1)$$

Multiplying both sides of (4.1) by the positive number x_1, we get

$$x_1^2 < x_1 x_2 \qquad \ldots (4.2)$$

Again multiplying both sides of (4.1) by the positive number x_2, we get

$$x_1 x_2 < x_2^2. \qquad \ldots (4.3)$$

From (4.2) and (4.3), we get

$$x_1^2 < x_2^2$$

i.e. $$f(x_1) < f(x_2)$$

Therefore, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
Hence, $f(x) = x^2$ is an increasing function in $(0, \infty)$.
The graph of $f(x) = x^2$ in $(0, \infty)$ is shown in Fig. 4.6.

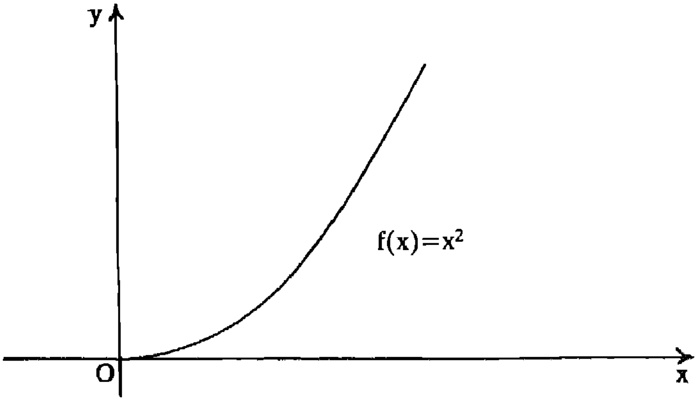


Fig. 4.6

Example 4.10

Show that $f(x) = x^2$ is a decreasing function in $(-\infty, 0)$.

Solution

Let $x_1, x_2 \in (-\infty, 0)$ and let $x_1 < x_2$.
Now $$x_1 < x_2 \qquad \ldots (4.4)$$
x_1, x_2 are negative numbers.
Multiplying both sides of (4.4) by the negative number x_1, we get

$$x_1^2 > x_1 x_2 \qquad \ldots (4.5)$$

Again multiplying both sides of (4.1) by the negative number x_2, we get

$$x_1 x_2 > x_2^2 \qquad \ldots (4.6)$$

From (4.5) and (4.6), we get

$$x_1^2 > x_2^2$$

i.e. $$f(x_1) > f(x_2)$$

Therefore, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Therefore, $f(x) = x^2$ is a decreasing function in $(-\infty, 0)$

The derivative of the function f is often very useful to determine the increasing or decreasing nature of f in an interval.

We state the following theorem :

Theorem 4 1

A differentiable real function $f(x)$ is increasing on an open interval I if and only if $f'(x) > 0$ for all x in I.

Explanation

If $f'(x)$ is positive then $f(x+h) - f(x)$ and h have the same sign for small values of h. When h is positive $x + h > x \Rightarrow f(x+h) > f(x)$ and when h is negative $x + h < x \Rightarrow f(x+h) < f(x)$.

This means $f(x)$ is an increasing function as shown in Fig. 4 7.

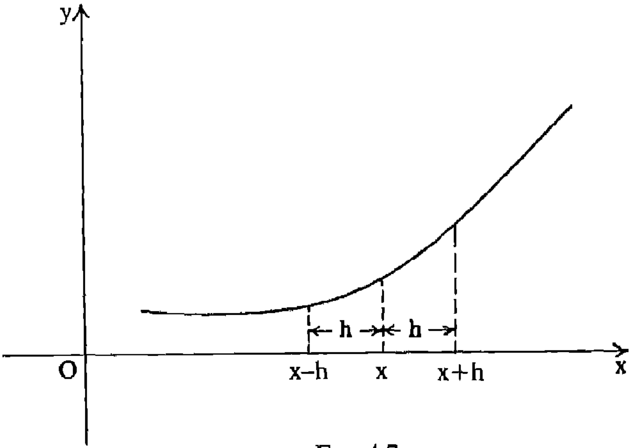


Fig 4 7

On similar lines, we have the following theorem :

Theorem 4.2

A differentiable real function is decreasing on an interval I if $f'(x) < 0$ for all x in I.

Example 4.11

Find the intervals in which the function

$f(x) = 2x^3 - 3x^2 - 36x + 7$

is (a) increasing (b) decreasing.

Solution

$f(x) = 2x^3 - 3x^2 - 36x + 7$

$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$

$f'(x) = 0$ gives $x^2 - x - 6 = 0$, i.e. $(x-3)(x+2) = 0$,

i.e. $x = 3$ or $x = -2$.

The points $x = -2$ and $x = 3$ divide the real line into 3 disjoint intervals, namely, $(-\infty, -2)$, $(-2, 3)$, $(3, \infty)$.

In the interval $(-\infty, -2)$, $f'(x)$ is positive. Therefore, $f(x)$ is increasing in this interval. In the interval $(-2, 3)$, $f'(x)$ is positive. Therefore, $f(x)$ is increasing in this interval.

In the interval $(3, \infty)$, $f(x)$ is positive. Therefore, $f(x)$ is increasing in this interval

Example 4 12

Prove that the exponential function e^x is increasing.

Remark

When the interval is not mentioned, we must prove that the function is increasing in its domain, in this case, **R**

Solution

We know that

$$\frac{d}{dx}(e^x) = e^x$$

We also know that when x is positive,

e^x is positive, because

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots \geq 1.$$

When x is negative,

$$e^x = \frac{1}{e^{-x}} = \frac{1}{\text{a positive quantity}} = \text{a positive quantity}$$

When x is zero, e^x is 1 and this is also positive.

Thus, e^x is positive for all values of x.

Thus, $\frac{d}{dx}(e^x)$ takes only positive values.

Therefore, e^x is an increasing function.

Example 4.13

Prove that the function $\sin x$ is increasing in the interval $(0, \frac{\pi}{2})$ and decreasing in the interval $(\frac{\pi}{2}, \pi)$.

Solution

$$\frac{d}{dx}(\sin x) = \cos x.$$

We know that $\cos x$ is positive in $(0, \frac{\pi}{2})$ and negative in $(\frac{\pi}{2}, \pi)$.

Therefore, $\sin x$ is increasing in $(0, \frac{\pi}{2})$ and decreasing in $(\frac{\pi}{2}, \pi)$.

Remark

On the interval $(0, \pi)$, $\sin x$ is neither increasing nor decreasing.

EXERCISE 4.4

1. Without using the derivative show that $f(x) = x^2$ for $x < 0$ is a decreasing function.
2. Show that $f(x) = x^2$ for all $x \in \mathbf{R}$ is neither increasing nor decreasing, without using the derivative.
3. Prove that $f(x) = ax + b$, where a and b are constants and $a > 0$ is an increasing function for all real values of x, without using the derivative.
4. Find the intervals in which the following functions are increasing or decreasing.
   - (a) $x^2 + 2x - 5$
   - (b) $10 - 6x - 2x^2$
   - (c) $-2x^3 - 9x^2 - 12x + 1$
   - (d) $6 - 9x - x^2$
   - (e) $(x+1)^3.(x-3)^3$
5. Prove that the logarithmic function is increasing wherever it is defined.
6. Prove that the function $x^3 - 3x^2 + 3x - 100$ is increasing on R.
7. Prove that the function $x^2 - x + 1$ is neither increasing nor decreasing on $(0, 1)$.
8. Which of the following functions are decreasing on $(0, \frac{\pi}{2})$?

   (a) $\cos x$ (b) $\cos 2x$ (c) $\cos 3x$ (d) $\tan x$.

9. On which of the following intervals is the function $x^{100}+\sin x-1$ increasing ?

   (a) $(-1, 1)$ (b) $(0, 1)$ (c) $(\frac{\pi}{2}, \pi)$ (d) $(0, \frac{\pi}{2})$

10. Find the least value of a such that the function x^2+ax+1 is increasing on $[1, 2]$.

11. Let I be any interval disjoint from $(-1, 1)$. Prove that the function $x+\frac{1}{x}$ is increasing on I.

12. Prove that the function $\log \sin x$ is increasing on $(0, \frac{\pi}{2})$ and decreasing on $(\frac{\pi}{2}, \pi)$.

4.5 Maxima and Minima

In this section we apply differentiation to calculate the maximum or minimum values of functions

Let us consider the following three problems that arise in practical situations :

Problem 1

A company finds that if it produces less, its profit is also less. If it produces too much, then it is unable to sell the commodity and therefore, there is loss. It wants to make the maximum profit. It finds that profit is given by the equation $p(x)=41-24x-18x^2$ where x is the quantity of production. What is the maximum profit that the company can make ? How much quantity should it produce to get this maximum profit ?

Problem 2

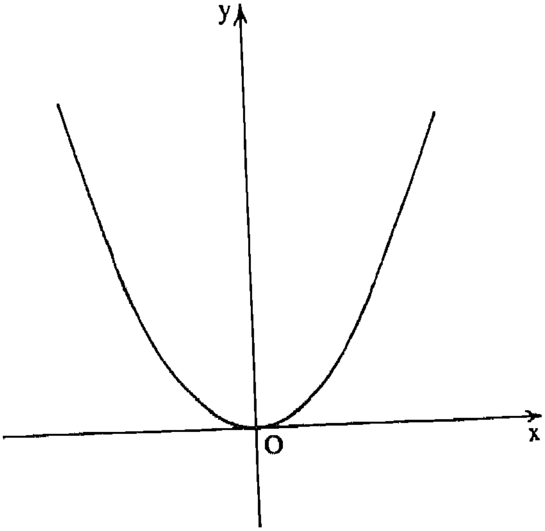
A jet of an enemy is flying along the curve $y=x^2+2$. A soldier, placed at the point $(3, 2)$, wants to shoot it when it is nearest to him. What is the nearest distance ?

Problem 3

What is the maximum height reached by a ball moving upwards under the formula $s=at-bt^2$, where a and b are constants.

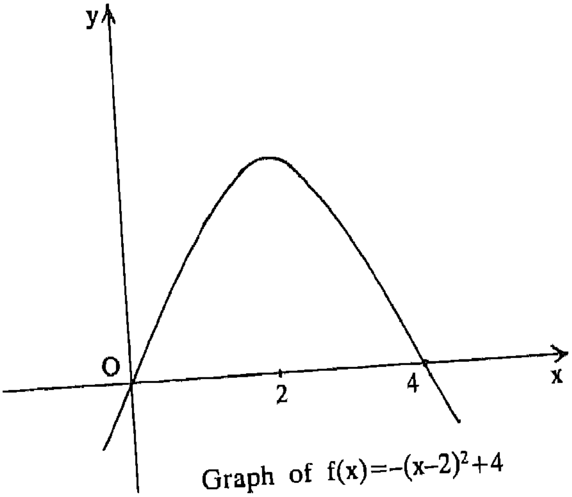
In these three problems, there is something common We want to find the maximum or minimum values of given functions in each of them.

In the case of some functions it is not difficult to find the maximum or minimum values of the functions without using calculus. For example the function $f(x)=x^2$ has only minimum value 0 in **R** (Fig. 4.8), where as the function $f(x)=-(x-2)^2+4$ has only the maximum value 4 in **R** (Fig. 4.9). The function $f(x)=x^3$ has neither any maximum value nor any minimum value in **R** (Fig 4.10)



Graph of $f(x)=x^2$

Fig. 4 8



Graph of $f(x)=-(x-2)^2+4$

Fig. 4 9

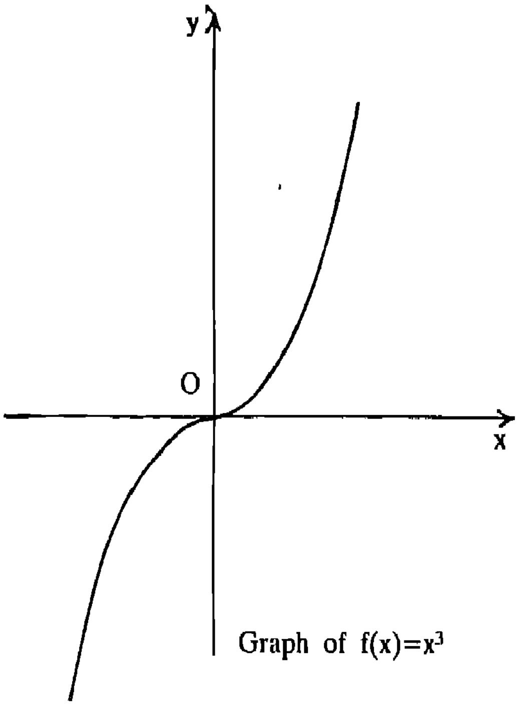


Fig. 4.10

Though the meaning of maximum value or minimum value is intuitively clear, let us define them mathematically.

Definition

A function $f(x)$ is said to have a maximum value in an interval I at x_0, if x_0 is in I and if $f(x_0) \geq f(x)$ for all x in I. The number $f(x_0)$ is called the maximum value of $f(x)$ in I and x_0 is called a (point of) maximum of $f(x)$ in I.

We can have a similar definition for the minimum value of a function.

Definition

A function $f(x)$ is said to have a minimum value in an interval I at x_0, if x_0 is in I and if $f(x_0) \leq f(x)$ for all x in I. The number $f(x_0)$ is called the minimum value of $f(x)$ in I and x_0 is called a point of minimum of $f(x)$ in I.

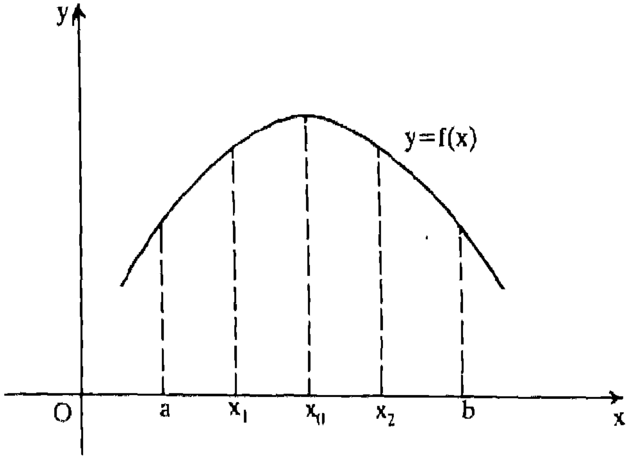


Fig. 4.11

In Fig. 4.11, $x=x_0$ is a point of maximum of $f(x)$ in the interval (a, b). In Fig. 4.12, $x=x_0$ is a point of minimum of $g(x)$ in the interval (a, b).

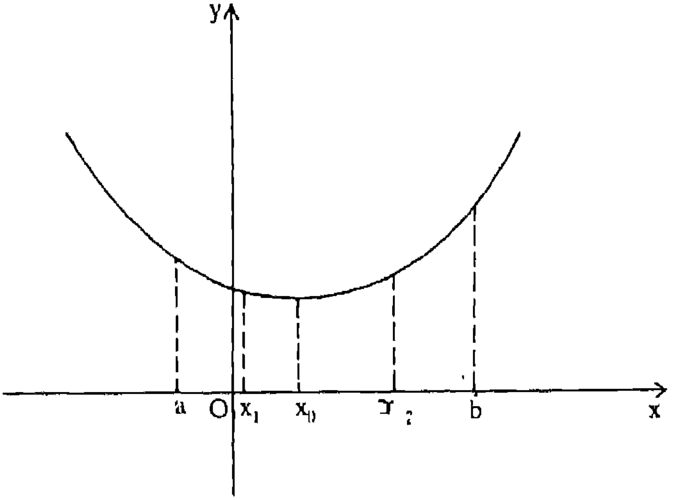


Fig. 4.12

x_0 is a point of maximum	x_0 is a point of minimum
$f(x_0) > f(x_1)$	$g(x_0) < g(x_1)$
$f(x_0) > f(x_2)$	$g(x_0) < g(x_2)$

Note : The function $f(x) = x$ does not have a maximum or minimum in the open interval (0,1), whereas according to a theorem, every continuous function on a closed interval, has a maximum and a minimum. For example the same function $f(x) = x$ defined in the closed interval [0,1] has the maximum value 1 and minimum value 0.

Example 4 13

Find the maximum/minimum values of $f(x) = 4x^2 - 4x + 11$, for any x in **R**.

Solution

$$f(x) = 4x^2 - 4x + 11$$
$$= (2x-1)^2 + 10$$

$(2x-1)^2$ is non-negative for all $x \in$ **R** and has least value 0.

When $2x - 1 = 0$, then $x = \frac{1}{2}$

Therefore, $f(x)$ has the minimum value 10 at $x = \frac{1}{2}$.

It has no maximum value in **R**.

Example 4.14

Find the maximum/minimum values of sin $(2x+3)$ in **R**.

Solution

We know that the sine function, sin θ, has the maximum value 1 and the minimum value -1 in **R** Hence, maximum value of sin $(2x+3)$ is 1 and minimum value is -1.

Look at the graph of the function $\varphi(x)$ in Fig. 4.13.

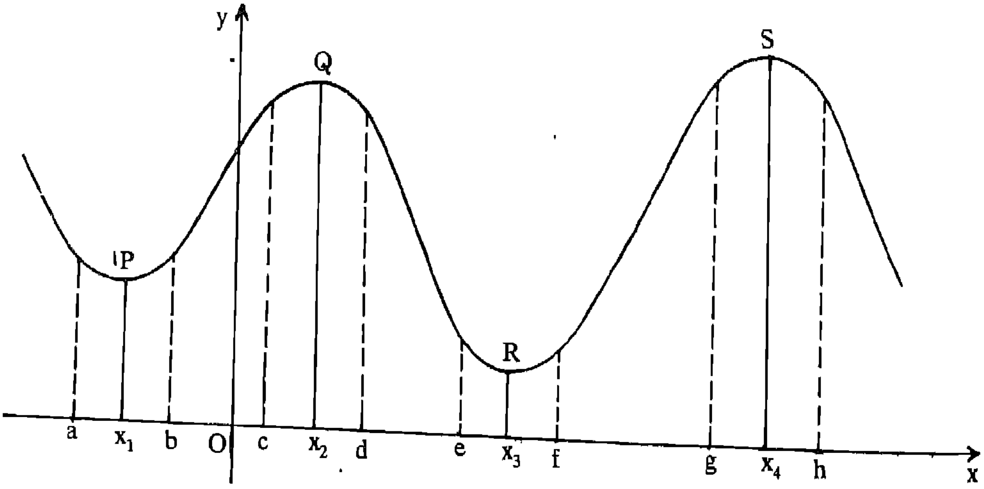


Fig. 4.13

The points P, Q, R, S are special points on the graph. The graph takes a 'turn' at each of these points. You can understand intuitively why we say that the graph takes 'turn' at each of the points. These points may be called 'the turning points' of the function.

Let us consider the point P corresponding to $x=x_1$.

Let (a, b) be a small interval around x_1 containing x_1. If we consider $\varphi(x)$ in the interval (a, b), we find that x_1, which is a point of minimum and $\varphi(x_1)$ is the minimum value of $\varphi(x)$ in this interval.

Similarly let us consider the point R on the graph corresponding to $x=x_3$. Let us again consider a small interval (e, f) around x_3 containing x_3. If we consider $\varphi(x)$ in the interval (e, f), we find that x_3 is a point of minimum and $\varphi(x_3)$ is the minimum value of $\varphi(x)$ in this interval

We say the point $x = x_1$ (corresponding to P) is a (point of) *local minimum* and the local minimum value of $\varphi(x)$ is $\varphi(x_1)$. Similarly $x = x_3$ (corresponding to R) is also a point of local minimum and the *local minimum value* is $\varphi(x_3)$.

In the same fashion, we call the point $x=x_2$ (corresponding to Q) as a point of *local maximum* and the local maximum value at this point is $\varphi(x_2)$. As you can see in the graph the point $x=x_4$ is also a point of local maximum.

Definition

Let f be a real function and let x_0 be an interior point in the domain of f. We say that x_0 is a local maximum of f (or a point of local maximum of f or simply a maximum of f), if there is an open interval containing x_0 such that $f(x_0) > f(x)$ for every x in that open interval.

Definition

Let f be a real function and x_0 be an interior point, in the domain of f. We say that x_0 is a local minimum of f (or a point of local minimum of f or simply a minimum of f) if there is an open interval containing x_0 such that $f(x_0) < f(x)$ for every x in that open interval.

How do we find the points of local minima or local maxima of a given function. Let us examine this.

If x_0 is a point of local maximum of $f(x)$ then the graph of $f(x)$ around x_0 will be as in Fig. 4.14.

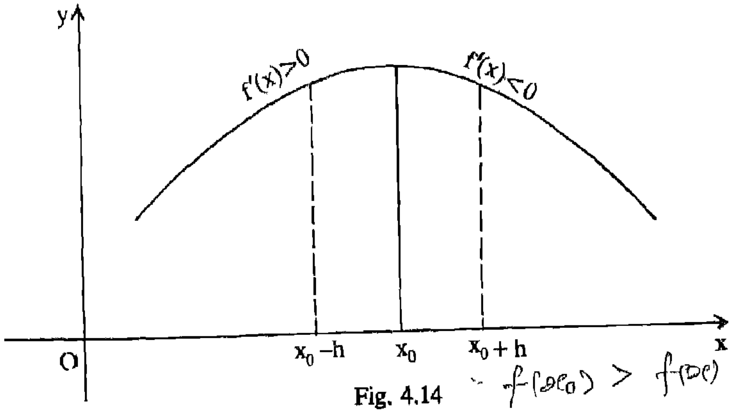


Fig. 4.14

$\therefore f(x_0) > f(x)$ for all $x \in$

As you can see $f(x)$ is increasing in the interval $(x_0 - h, x_0)$ and decreasing in the interval $(x_0, x_0 + h)$. So in $(x_0 - h, x_0)$, $f'(x) > 0$ and in $(x_0, x_0 + h)$, $f'(x) < 0$. This suggests that $f'(x_0)$ must be zero. Similarly if x_0 is a point of local maximum of $f(x)$, then the graph of $f(x)$ around x_0 will be as in Fig. 4.15.

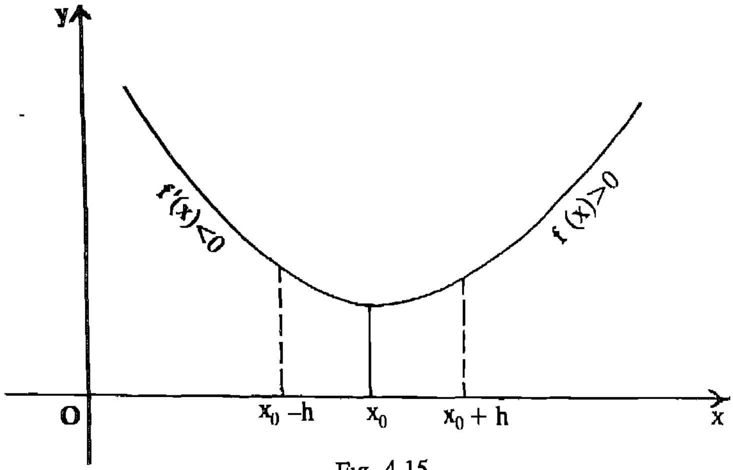


Fig. 4.15

As you can see $f(x)$ is decreasing in $(x - h, x_0)$ and increasing, in $(x_0, x_0 + h)$. So $f'(x) < 0$ in $(x_0 - h, x_0)$ and > in $(x_0, x_0 + h)$ suggesting that $f'(x_0)$ must be zero.

We have the following two theorems which we state without proof :

Theorem 4.3

Let f be a differentiable function. Then f' vanishes at every local maximum and at every local minimum.

Note

(i) f' vanishes at every local maximum or local minimum. But the converse is not true, that is, every point at which f' vanishes need not be a local maximum or minimum. For example if $f(x) = x^3$, $f'(x) = 3x^2$, and so $f'(0) = 0$; but 0 is neither a local minimum nor a local maximum.

(ii) $f'(x_0)$ gives the slope of the tangent to the curve given by $f(x)$ at the point x_0. So when x_0 is a point of local minimum or local maximum, the tangent at x_0 is parallel to the x-axis.

Remark

This theorem helps us to find local maximum and local minimum of functions. A search for these points may be confined to the points at which the derivatives vanish.

The values of the function corresponding to points of local minimum or points of local maximum are called the extreme values of the functions.

The derivative $f'(x)$ gives us the points of local minima or points of local maxima How do we distinguish whether the point x_0 satisfying $f'(x) = 0$ is a point of local maximum or a point of local minimum ? We have seen that if x_0 is a point of local maximum, $f'(x) > 0$ at a nearby point to the left of x_0 and 0 at a nearby point to the right of x_0. On the other hand if x_0 is a point of local minimum, $f'(x) < O$ at a point immediately to the left of x_0 and 0 at a point immediately to the right of x_0. Thus, we have the following working rule for finding the points of local maxima or points of local minima.

Theorem 4 4 (First Derivative Test)

Let $f(x)$ be a differentiable function on I and let $x_0 \in I$. Then

1. x_0 is a point of local maximum of $f(x)$ if
   (i) $f'(x_0) = 0$
   (ii) $f'(x) > 0$ at every point close to and to the left of x_0; and $f'(x) < 0$ at every point close to and to the right of x_0
2. x_0 is a point of local minimum of $f(x)$ if
   (i) $f'(x_0) = 0$
   (ii) $f'(x) < 0$ at every point close to and to the left of x_0; and $f'(x) > 0$ at every point close to and to the right of x_0.
3. If $f'(x_0) = 0$, but $f'(x)$ does not change sign as x increases through x_0, then x_0 is neither a point of local minimum nor a point of local maximum.

Remark

If $f'(x_0) = 0$ and x_0 is neither a point of local minimum nor a point of local maximum, then x_0 is called a point of inflection.

Example 4.15

Find all the (local) maxima and minima of the function

$$f(x) = x^3 - 12x.$$

Solution

$f(x) = x^3 - 12x$

$\therefore f'(x) = 3x^2 - 12 = 3\ (x-2)\ (x+2)$

f' vanishes at the points $x = 2$ and $x = -2$.

We have to examine whether these points are local maximum or local minimum or neither of them.

Let us take $x = 1.9$ which is to the left of $x = 2$ and $x = 2.1$ which is to the right of 2 and find $f'(x)$ at these points.

$f'(1.9) = 3\ (1.9-2)\ (1.9+2)$ which is negative

$f'(2.1) = 3(2.1-2)\ (2.1+2)$ which is positive.

Thus from the first derivative test $x = 2$ is a local minima.

Again let us consider the points $x = -2.1$ which is to the left of $x = -2$ and $x = -1.9$ which is to the right of $x = -2$

$f'(-2.1) = 3(-2.1 - 2)(-2.1 + 2)$ which is positive

$f'(-1.9) = 3(-1.9 - 2)(-1.9 + 2)$ which is negative

From the first derivative test $x = -2$ is a point of local maximum.

Sign of $f'(x)$			
Point $x = 2$		Point $x = -2$	
Left of 2	Right of 2	Left of -2	Right of -2
negative	positive	positive	negative
Local minimum		Local maximum	

Example 4.16

Find all the points of local maxima and minima of the function $f(x) = x^3 - 6x^2 + 12x - 8$.

Solution

$f(x) = x^3 - 6x^2 + 12x - 8$

$\therefore f'(x) = 3x^2 - 12x + 12 = 3(x - 2)^2$

$f'(x) = 0$ gives $x = 2$.

Let us see whether $x = 2$ is a point of local maximum or minimum. Let us take $x = 1.9$ to the left of the point $x = 2$ and $x = 2.1$ to the right of the point

$f'(1.9) = 3(1.9 - 2)^2$ which is a positive number.

$f'(2.1) = (2.1 - 2)^2$ which is a positive number

$\therefore f'(x)$ does not change sign as x increases through $x = 2$.

Hence, $x = 2$ is neither a point of local maximum nor a point of local minimum. (It is a point of inflexion).

Second Derivative Test

The first derivative test helps us in finding the local maximum or local minima, but it takes time to verify how $f'(x)$ is changing sign as x passes through the points given by $f'(x) = 0$. We have another test known as the second derivative test which enables us to find the points of local maxima or local minima.

Consider a point x_0 such that $f'(x_0) = 0$ and $f''(x_0) < 0$. This suggests that f' is decreasing at x_0, because its derivative is negative. Therefore, $f'(x)$ is positive to the left of x_0, and is negative to the right of x_0, in a small interval around x_0. This in turn implies that in this interval, $f(x)$ is increasing upto x_0' and then decreasing. Therefore, x_0 is a point of local maximum.

Thus, if $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is a point of local maximum. See Fig. 4.16.

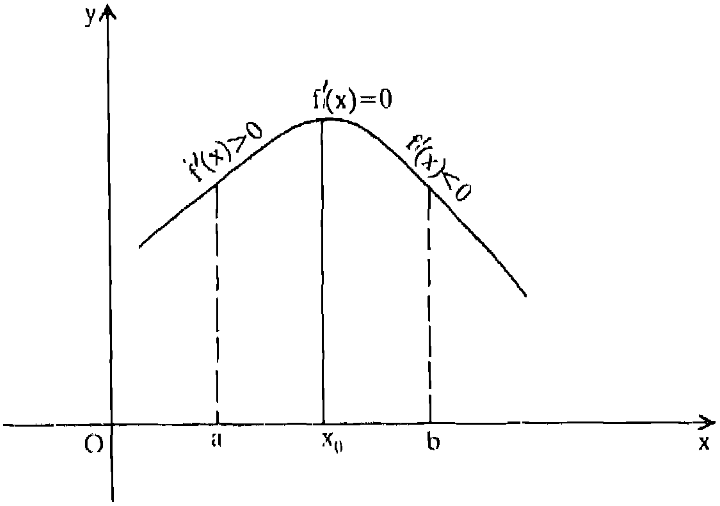


Fig 4.16

Similarly, if $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a point of local minimum. See Fig. 4.17.

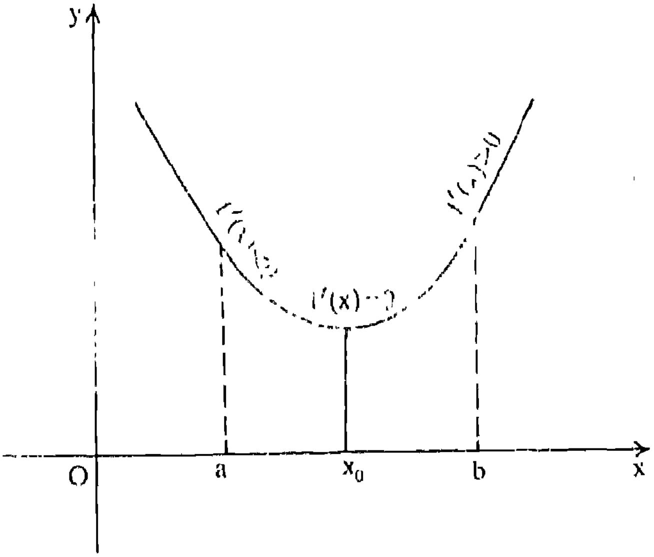


Fig 4 17

Theorem 4.5 : (Second derivative test)

Let $f(x)$ be a differentiable function on I and let $x_0 \in I$.
Let $f''(x)$ be continuous at x_0. Then

1. x_0 is a local maximum if both $f'(x_0) = 0$ and $f''(x_0) < 0$.
2. x_0 is a local minimum if both $f'(x_0) = 0$ and $f''(x_0) > 0$.

Note : (i) The second derivative test fails if $f''(x_0) = 0$. In that case we have to go back to the first derivative test to find whether x_0 is a point of local maximum/minimum or not.
(ii) If $f''(x_0) = 0$ and x_0 is not a point of local maximum or local minimum, then x_0 is a point of 'inflection'.

We have the following rule for finding the local minimum or local maximum of a function f:

Step 1. Find the points at which f' vanishes
Step 2. At each of these points, find the sign of f''
Step 3. If $f'(x) = 0$ and $f''(x) > 0$, conclude that x is a local minimum
If $f'(x) = 0$ and $f''(x) < 0$, conclude that x is a local maximum.

Example 4.17

Find all the local maxima or minima of the function
$f(x) = x^3 - 12x$.

Solution

We solved the same example using the first derivative test.
Let us solve this example again using the second derivative test and see whether we are getting the same solution.

$f(x) = x^3 - 12x$
$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2)$
$f''(x) = 6x$
$f'(x) = 0$ gives the points $x = 2$ and $x = -2$.

Now $f''(2) = 6 \times 2 = 12$ (Positive) and
$f''(-2) = -12$ (Negative).

Hence, from the second derivative test, we conclude that
$x = 2$ is a point of local minimum and
$x = -2$ is a point of local maximum.

Also we got the same results from the first derivative test.

Example 4.18

Find all the points of local maxima and minima of the function $f(x) = x^3 - 6x^2 + 12x - 8$.

Solution

$f(x) = x^3 - 6x^2 + 12x - 8$

$f'(x) = 3x^2 - 12x + 12 = 3\ (x-2)^2$

$f''(x) = 6(x-2)$

$f'(x) = 0$ gives $x = 2$.

$f''(2) = 0$.

Hence, the second derivative test fails here and we have to go back to the first derivative test. You have already seen that this point $x = 2$ is neither a point of local maximum nor a point of local minimum after applying the first derivative test. So $x = 2$ is a point of inflexion.

Example 4.19

Find all the maxima and minima of the sine function.

Solution

Let $f(x) = \sin x$.

Then $f'(x) = \cos x$.

The points at which f' vanishes are

$$\frac{\pi}{2}, \frac{\pi}{2} \pm \pi, \frac{\pi}{2} \pm 2\pi, \ldots$$

Next, $f''(x) = -\sin x$.

$$f''\left(\frac{\pi}{2}\right) = -\sin \frac{\pi}{2} = -1 \text{ is negative.}$$

$$f''\left(\frac{\pi}{2} \pm \pi\right) = -\sin\left(\frac{\pi}{2}\right) = 1 \text{ is positive.}$$

Generally, $f''\left(\frac{\pi}{2} \pm n\pi\right)$ is negative if n is an even integer and positive if n is an odd integer.

Therefore, for the function $\sin x$

$$\frac{\pi}{2}, \frac{\pi}{2} \pm 2\pi, \frac{\pi}{2} \pm 4\pi, \ldots \text{ are local minima}$$

and $$\frac{\pi}{2} \pm \pi, \frac{\pi}{2} \pm 3\pi, \ldots \text{ are local maxima.}$$

Maximum and Minimum Values in a Closed Interval

Consider the function $f(x)$ defined in the closed interval $[a, b]$ whose graph is shown in Fig. 4.18.

What is the maximum and minimum values of $f(x)$ in $[a, b]$. Evidently the maximum value is $f(a)$ and the minimum value is $f(b)$. Recall that in the interval $[a, b]$, $x = x_1$ is a point of local minimum and the local minimum value of f is $f(x_1)$.

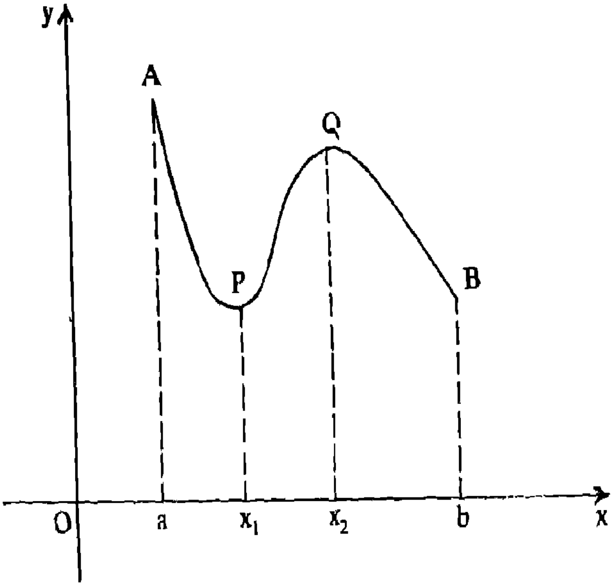


Fig. 4.18

Similarly $x = x_2$ is a point of local maximum and the local maximum value of f is $f(x_2)$. But neither $f(x_2)$ is the maximum value of $f(x)$ in $[a, b]$ nor $f(x_1)$ is the minimum value of $f(x)$ in $[a, b]$. We say, in the interval $[a, b]$, $f(a)$ is the *absolute* minimum value of $f(x)$ or simply maximum value of $f(x)$ and $f(b)$ is the *absolute* minimum value of $f(x)$ or simply minimum value of $f(x)$. The distinction between absolute maximum value and the local maximum (minimum) value must be clear now. It is possible that in a given interval, the absolute maximum value of the function and the local maximum value may be the same.

The following theorems (Proofs are not given) help us to find the absolute maximum and minimum value of a function on an interval I.

Theorem 4.6

Let f be a continuous function on an interval I of the type $a \leq x \leq b$. Then f has the maximum value and attains it at least once in I. Also f has the minimum value and attains it at least once in I.

This theorem asserts that f has the maximum value and the minimum value in a given closed interval.

Theorem 4.7

If a differentiable function f attains its maximum value at an interior point x_0 of its domain then $f'(x_0) = 0$.

Theorem 4.8

If a differentiable function f attains its minimum value at an interior point x_0 of its domain, then $f'(x_0) = 0$.

How do these theorems help us in solving the practical problems. They say that the search for interior points where the (absolute) maximum or minimum values are taken by the function may be confined to those points where the derivative takes the value 0. In other words, it makes this search simpler. It leads to the following rule for finding the maximum or minimum values of a function in a given interval.

Step 1 : Find all the points where f' takes the value zero
Step 2 : Take the end values of the interval
Step 3 : At all these points calculate the value of f
Step 4 : Take the maximum and minimum values out of the values calculated in step 3.

Example 4.20

Find the maximum value of $x^{50} - x^{20}$ on the interval [0, 1]. Find also its minimum value on this interval.

Solution

Let $f(x) = x^{50} - x^{20}$.
Then $f'(x) = 50x^{49} - 20x^{19}$.
Our first step is to find all those points where f' takes the value zero.
If $f'(x) = 0$, then $50x^{49} = 20x^{19}$; then
either $x = 0$ or $50x^{30} = 20$.

If $50x^{30} = 20$, then $x = \left(\frac{2}{5}\right)^{\frac{1}{30}}$

Therefore, $x = 0$ and $x = \left(\frac{2}{5}\right)^{\frac{1}{30}}$, are the only two points where f' takes the value zero.

With these two points, we take the two end points 0 and 1 of the interval. Totally we get only three points. At these three points, we calculate the value of f.

$f(0) = 0$
$f(1) = 0$

$$f\left[\left(\frac{2}{5}\right)^{\frac{1}{30}}\right]=\left(\frac{2}{5}\right)^{\frac{50}{30}}-\left(\frac{2}{5}\right)^{\frac{20}{30}}$$

$$=\left(\frac{2}{5}\right)^{\frac{5}{3}}-\left(\frac{2}{5}\right)^{\frac{2}{3}}$$

This is a negative quantity.

Of these three values, the maximum value is 0, and the minimum value is

$$\left(\frac{2}{5}\right)^{\frac{1}{3}}-\left(\frac{2}{5}\right)^{\frac{1}{3}}.$$

Therefore, our answer is :

The maximum value of $f(x)$ on (0, 1) is 0.

The minimum value of $f(x)$ on (0, 1) is $\left(\frac{2}{5}\right)^{\frac{5}{3}}-\left(\frac{2}{5}\right)^{\frac{2}{3}}$

Example 4.21

A jet of an enemy is flying along the curve $y=x^2+2$.

A soldier is placed at the point (3, 2). What is the nearest distance between the soldier and the jet ?

Solution

For each value of x, the jets' position is (x, x^2+2).

Let $f(x)$ be the square of the distance between this position and the soldier. Then

$f(x)=$ Square of distance between (x, x^2+2) and $(3, 2)$

$=(x-3)^2+(x^2+2-2)^2$

$=(x-3)^2+x^4$

First, we find the minimum value of $f(x)$.

For this purpose, we differentiate :

$$f'(x)=2(x-3)+4x^3$$
$$=2(x-1)(2x^2+2x+3).$$

The only point at which f' takes the value zero is $x=1$. There are no real roots for $2x^2+2x+3=0$.

(There are no end points ot interval to be added to this set).

The value of f at this point is

$f(1)=(1-3)^2+1^4=4+1=5.$

Thus, 5 is the minimum value of $f(x)$.

(Why is this not the maximum value ? We check that $f(0) = 9 > 5$.
Therefore, 5 cannot be the maximum value. But then what is the maximum value. It does not exist in this example).

Remember that $f(x)$ is the square of the required distance. Therefore, the minimum distance required is $\sqrt{5}$

EXERCISE 4.5

1. Find the maximum or minimum values, if any, of the following functions without using the derivatives :
   - (i) $(2x-1)^2+3$
   - (ii) $-(x-1)^2+10$
   - (iii) $9x^2+12x+2$
   - (iv) x^3+1
   - (v) $|x+2|$
   - (vi) $-|x+1|+3$
   - (vii) $\sin 2x+5$
   - (viii) $|\sin 4x+3|$
   - (ix) $\sin \sin x$
2. Find the local maxima or local minima, if any, of the following functions, using the first derivative test only. Find also the local maximum or local minimum values, as the case may be :
   - (i) The constant function α
   - (ii) x^2
   - (iii) x^3-3x
   - (iv) $\cos x,\ 0 < x < \pi$
   - (v) $\sin 2x,\ 0 < x < \pi$
   - (vi) $\sin x+\cos x, 0 < x < \frac{\pi}{2}$
   - (vii) $\sin x-\cos x,\ 0 < x < 2\pi$
   - (viii) $x^3-6x^2+9x+15$
   - (ix) $(x-1)(x+2)^2$
   - (x) $\frac{x}{2}+\frac{2}{x},\ x>0$
   - (xi) $\frac{1}{x^2+2}$
   - (xii) $3\sqrt{x^2}-4,\ x>0$
   - (xiii) $x\sqrt{1-x},\ x>0$
   - (xiv) $\sin^4 x+\cos^4 x,\ 0 < x \leq \frac{\pi}{2}$
   - (xv) $\sin 2x-x,\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
   - (xvi) $(x-3)^4$
   - (xvii) $x^3(x-1)^2$
   - (xviii) $x^3(2x-1)^3$
   - (xix) $-(x-1)^3(x+1)^2$
3. Find the local maxima or local minima, if any, of the functions, given in exercise 2 above, using the second derivative test. Indicate if the second derivative test fails.
4. Prove that the following functions do not have maxima or minima
   - (i) e^x
   - (ii) $\log x$

(iii) $x+2$
(iv) x^3+x^2+x+1

5. Find the absolute value and the absolute minimum value of the following functions in the given intervals
   (i) $f(x)=x^3$ in $[-2, 2]$
   (ii) $f(x)=(x-1)^2+3$ in $[-3, 1]$
   (iii) $f(x)=(\frac{1}{2}-x)^2+x^3$ in $[-2, 2.5]$
   (iv) $f(x)=\sin x+\cos x$ in $[0, \pi]$
   (v) $f(x)=4x-\frac{1}{2}x^2$ in $[-2, 4.5]$

6. Find the maximum profit that a company can make, if the profit function is given by $p(x)=41-24x-18x^2$.

7. Find both the maximum value and the minimum value of $3x^4-8x^3+12x^2-48x+25$ on the interval $[0, 3]$.

8. Find both the maximum value and the minimum value of $3x^4-8x^3+12x^2-48x+1$ on the interval $[1, 4]$

9. At what points in the interval $[0, 2\pi]$ does the function $\sin 2x$ attain its maximum value ?

10. What is the maximum value of the function $\sin x+\cos x$?

11. Find the maximum value of $2x^3-24x+107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, -1]$

12. It is given that at $x=1$, the function x^4-62x^2+ax+9 attains its maximum value, on the interval $[0, 2]$. Find the value of a.

13. Find the maximum and minimum value of $x+\sin 2x$ on $[0, 2\pi]$

14. Find the numbers whose sum is 24 and whose product is as large as possible.

15. Find two positive numbers x and y such that $x+y=60$ and xy^3 is maximum.

16. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a minimum.

17. Find two positive numbers whose sum is 16 and the sum of whose cubes is maximum.

18. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flops to form the box. What should be the side of the square to be cut off, so that the volume of the box is the maximum possible.

19. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off squares from the corners and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum possible

20. Show that, of all the rectangles inscribed in a given fixed circle the square has the maximum area.

21. A beam of length l is supported at one end. If W is the uniform load per unit length, the bending moment M at a distance x from the end is given by

$$M=\frac{1}{2}\,lx-\frac{1}{2}\,Wx^2 .$$

Find the point on the beam at which the bending moment has the maximum value.

22. Of all the closed cylindrical cans (right circular), which enclose a given volume of 100 Cubic centimetres, which has the maximum surface area.
(*Hint* : $v=\pi r^2h$ and $S=2\pi r^2+2\pi rh$. Express S as a function of r).

23. A wire of length 28m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the lengths of the two pieces so that the combined area of the square and the circle is minimum.

24. The combined resistance R of two resistors R_1 and R_2 (R_1, $R_2>0$) is given by

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$$

If $R_1+R_2=C$ (a constant), show that the maximum resistance R is obtained by choosing $R_1=R_2$.

4.6 Rolle's Theorem

Consider the three functions :

1. $\sin x$
2. x^2-4x+3
3. $\sin x-\cos x$.

The first function, i.e. $\sin x$ vanishes (i.e. takes the value zero) at the points

$$0,\ \pm\pi,\ \pm 2\pi,\dots$$

Its derivative is $\cos x$

It vanishes at the points $\pm\frac{\pi}{2}\,,\ \pm\frac{3\pi}{2},\dots$

We note that between any two points where $\sin x$ vanishes, there is a point where its derivative $\cos x$ vanishes.

For instance, between 0 and π, there is $\frac{\pi}{2}$;

between π and 2π, there is $\frac{3\pi}{2}$; and so on.

The second function, i.e. x^2-4x+3, vanishes at $x=1$ and $x=3$.

Its derivative is $2x-4$. This derivative vanishes at $x=2$.

Here again 2 is between 1 and 3. Thus, between the two points where the function vanishes, there is a point where its derivative vanishes.

The third function, $\sin x - \cos x$ vanishes at the points

$$\frac{\pi}{4}, \frac{\pi}{4} \pm \pi, \frac{\pi}{4} \pm 2\pi, \ldots$$

Its derivative is $\cos x + \sin x$. It vanishes at the points.

$$\frac{\pi}{4} \pm \frac{\pi}{2}, \frac{\pi}{4} \pm \frac{3\pi}{2}, \ldots$$

Here again between $\frac{\pi}{4}$ and $\frac{\pi}{4} + \pi$, there is $\frac{\pi}{4} + \frac{\pi}{2}$ and so on

Between any two points where $\sin x - \cos x$ vanishes, there is a point where its derivative vanishes.

These observations in the above three example make us ask the question : Is the observed result accidental in these examples ? Or is it true in general that between any two points where $f(x)$ vanishes, there is at least one point where $f'(x)$ vanishes ?

Rolle's theorem asserts that the observed result is a general truth. We need not assume that f is defined on the whole **R**, as it happened in the above examples. Let $a < b$ be two points such that $f(x) = f(b) = 0$. Then it is true that there is a point c, $a < c < b$, such that $f'(c) = 0$ What all we have to assume is that f is differentiable in the open interval (a, b) and continuous in the closed interval $[a, b]$. The assumption of differentiability is reasonable because the conclusion is about the vanishing of the derivative This assumption implies that f is continuous in (a, b). It is reasonable to assume continuity at a and b also, because the hypothesis $f(a) = 0 = f(b)$ has no effect on the behaviour of f in (a, b), in the absence of this assumption.

Theorem 4.9 (*Rolle's theorem*)

Let f be a real function defined in the closed interval $[a, b]$ such that .

(i) $f(a) = f(b) = 0$

(ii) f is continuous in the closed interval $[a, b]$

(iii) $f(x)$ is differentiable in the open interval (a, b)

Then there is some point c in the open interval (a, b) such that $f'(c) = 0$.

Remark 1

Some authors state Rolle's Theorem, by relaxing condition (i) as $f(a) = f(b)$, without requiring them to be necessarily zero. This statement is also correct. In fact these two statements are equivalent This can be seen by taking $g(x) = f(x) - f(a)$ and observing that $g(a) = g(b) = 0$ and that $g'(x) = f'(x)$.

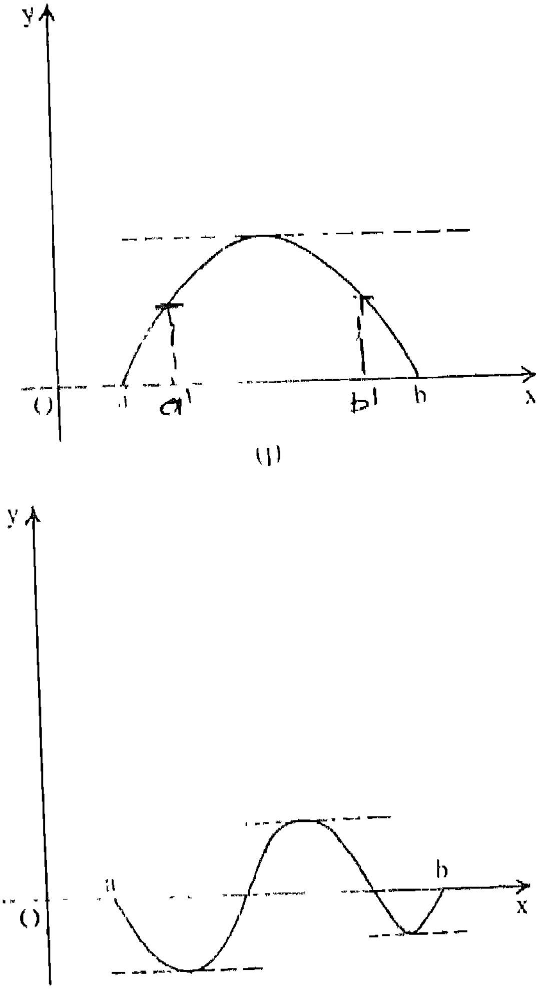
Note In view of the above remark, for verifying Rolle's theorem in specific problems, it is enough to ensure $f(a) = f(b)$, instead of requiring $f(a) = f(b) = 0$.

Remark 2

There can be more than one such c. Because f is continuous on $[a, b]$, it attains its maximum value at some point in $[a, b]$, as stated in the last section. If this point happens to be an interior point, then f' vanishes at that point, by the theorem of the last section. Are we not very near the conclusion of Rolle's Theorem ? Yes, except for the case when the maximum is attained at a or b We omit the further details of the proof

Geometric meaning of Rolle's Theorem

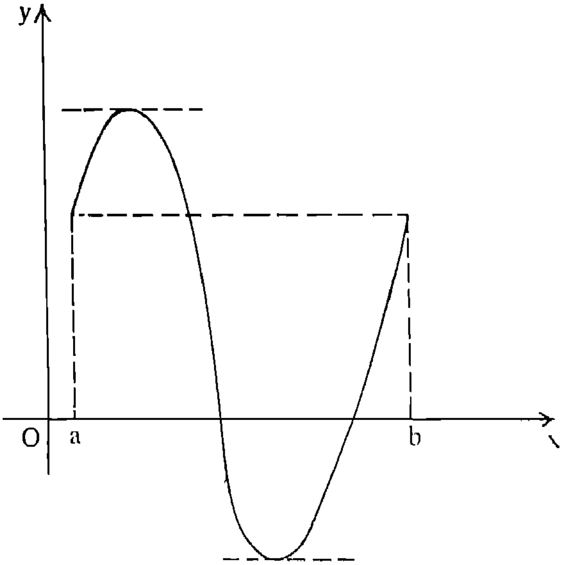
Under the assumptions of Rolle's theorem, the graph of f starts at $(a, 0)$ and ends at $(b, 0)$ as shown in Fig 4.19.



(i)

(ii)

Fig. 4 19



(iii)

Fig 4 19

The conclusion is that there is a point c between a and b, such that the tangent to the graph at $(c, f(c))$ is parallel to the x-axis

(Note $f'(c) = 0$ means the slope of the tangent at $(c, f(c))$ is zero).

This is believable because, if the graph is fixed, but the segment $[a, b]$ is moved upwards and downwards keeping it parallel to the x-axis, it must somewhere become a tangent to the curve, as indicated in the diagram You will learn the rigorous proof in higher classes

Now we solve some problems to understand Rolle's theorem more clearly

Example 4 22 For the function $\sin x - 1$ on $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$, verify the truth of Rolle's theorem.

Solution

Let $f(x) = \sin x - 1$

Then $f\left(\frac{\pi}{2}\right) = 1 - 1 = 0.$

and $f\left(\frac{5\pi}{2}\right) = 1 - 1 = 0$

The function $f(x)$ is differentiable everywhere (and so continuous).

Therefore, the conditions of Rolle's theorem hold good. To verify the conclusion, we first find that

$$f'(x) = \cos x.$$

We are now looking for a point c between $\frac{\pi}{2}$ and $\frac{5\pi}{2}$ such that $\cos c = 0$. We know $\cos\frac{\pi}{2}$ but we can't take $c = \frac{\pi}{2}$. This is because the c of Rolle's theorem has to be in the open interval $\left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$. We take $c = \frac{3\pi}{2}$ and verify that $\cos c = 0$.

Example 4.23

Let $f(x) = (x-1)(x-2)(x-3)$ on the interval $[1, 3]$. Prove that there is more than one c in $(1, 3)$ such that $f'(c) = 0$.

Solution

$$\begin{aligned} f'(x) &= (x-1)(x-2) + (x-2)(x-3) + (x-3)(x-1) \\ &= x^2 - 3x + 2 + x^2 - 5x + 6 + x^2 - 4x + 3 \\ &= 3x^2 - 12x + 11. \end{aligned}$$

The roots of $f'(x) = 0$ are $\frac{12 + \sqrt{144 - 132}}{6} = 2 \pm \frac{1}{\sqrt{3}}$.

Both these roots are in the open interval $(1, 3)$. Thus, there are two points c in $(0, 1)$ such that $f'(c) = 0$.

EXERCISE 4.6

Verify the conditions of Rolle's theorem in the following problems, 1 through 6. In each case, find a point in the interval where the derivative vanishes.

1. $x^2 - 1$ on $[-1, 1]$
2. $(x^2 - 1)(x - 2)$ on $[-1, 2]$
3. $\sin x - \sin 2x$ on $[0, \pi]$
4. $\log(x^2 + 2) - \log 3$ on $[-1, 1]$

5. e^{1-x^2} on $[-1, 1]$

6. $\sin x + \cos x - 1$ on $\left[0, \frac{\pi}{2}\right]$

7. It is given that for the function
$f(x) = x^3 - 6x^2 + ax + b$ on $[1, 3]$,
Rolle's theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$. Find the values of a and b.

8. At what points on the following curves, is the tangent parallel to the x-axis ?
   (a) $y = x^2$ on $[-2, 2]$
   (b) $y = \cos x - 1$ on $[0, 2]$

4.7 The Mean Value Theorem

In Rolle's theorem, we assumed that the end points of the graph were on the x-axis, and concluded that somewhere on the graph, the tangent is parallel to the x-axis.

We now improve this result, by saying that x-axis is not important here. We say that if the end points of the graph are on a line, then there is a point on the graph, where the tangent is parallel to that line. In other words, there is always a point on the graph, where the tangent is parallel to the line joining the end points of the graph.

Suppose we call the line joining any two points of the curve as a chord of the curve. Then our improved theorem reads as under : Given any chord of the graph of f, there is a point on the graph where the tangent is parallel to this chord as shown in Fig. 4.20.

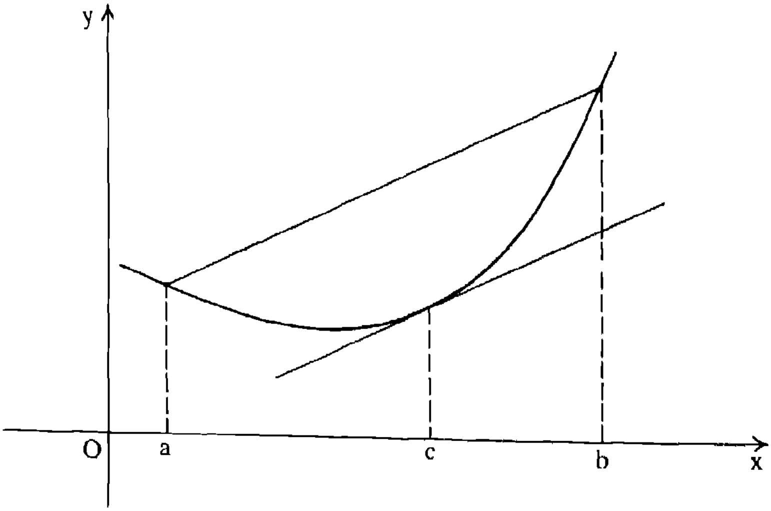


Fig. 4.20

In other words, to restate this theorem, we do some work using coordinate geometry

Let $(a, f(a))$ be two points on the graph of f. We may assume $a < b$. Then we assert the existence of c, $a < c < b$ such that the tangent at $(c, f(c))$ to the graph of f, is parallel to the chord joining $(a, f(a))$ and $(b, f(b))$.

We know that two lines are parallel if and only if they have the same slope.

The slope of the chord here is same as the slope of the line joining $(a, f(a))$ and $(b, f(b))$.

By a known formula, this is equal to

$$\frac{f(b)-f(a)}{b-a}.$$

On the other hand, the slope of the tangent at $(c, f(c))$ is $f'(c)$ (We have already seen this meaning of the derivative). Thus our c is to satisfy

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Now we are ready to state.

Theorem 4.10

(The Mean value theorem) : Let f be a real function, continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b). Then there is a point c in the open interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Remark

We shall not give proof of this result. We point out that in the particular case where $f(a) = f(b)$, the expression $\frac{f(b)-f(a)}{b-a}$ becomes zero. Thus when $f(a) = f(b)$, $f'(c) = 0$ where c is in (a, b).

Thus, Rolle's theorem becomes a particular case of the mean value theorem

Physical Meaning of the Mean Value Theorem

Imagine a car running on a straight road. At the time a let its position be $f(a)$. Afterwards, at the time b, let its position be $f(b)$. Then $f(b) - f(a)$ is the distance travelled and $b - a$ is the time taken Thus $\frac{f(b)-f(a)}{b-a}$ is the average speed of the car.

Now consider the speedometer of the car that shows the speed at any instant The speedometer reading is the value of f' at that time.

What the mean value theorem says is that at some point of time between a and b, the speedometer reading would have been equal to the average speed of the car

If the car has travelled 80 km in 2 hours, at some time the speed must have been exactly 40 km/h. We believe it because it cannot always be less than 40, nor can it be always greater than 40, and so the pointer has to cross the 40 km mark at some time

Remark

The word 'mean' means average. Here we are considering the average value $\dfrac{f(b)-f(a)}{b-a}$ and are asserting that it is one of the values taken by the derivative f'.

We provide some examples below to understand meanvalue theorem.

Example 4.24

Verify the truth of mean value theorem for the function $\sin x$ on the interval $\left[\dfrac{\pi}{2}, \dfrac{5\pi}{2}\right]$

Solution

$\sin x$ is differentiable everywhere. Therefore, the assumptions of mean value theorem hold good. On this interval, the mean value is

$$\frac{\sin\dfrac{5\pi}{2} - \sin\dfrac{\pi}{2}}{\dfrac{5\pi}{2} - \dfrac{\pi}{2}} = \frac{1-1}{2\pi} = 0.$$

We want to verify that the derivative of $\sin x$, namely $\cos x$, takes this mean value 0, at some point in the interval $\left(\dfrac{\pi}{2}, \dfrac{5\pi}{2}\right)$ This is easily verified because $\cos\dfrac{3\pi}{2} = 0$. (See Fig 4.21).

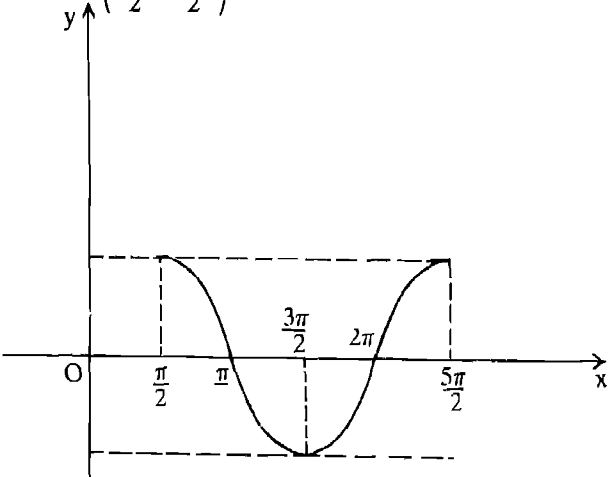


Fig 4.21

Example 4 25

On the curve $y=x^2$, find a point at which the tangent is parallel to the chord joining (0, 0) and (1, 1).

Solution

The slope of the chord is $\frac{1-0}{1-0}=1.$

The derivative is $\frac{dy}{dx}=2x.$

We want x such that $2x=1.$

Thus $x=\frac{1}{2}.$

We note that $\frac{1}{2}$ is in the open interval (0, 1), as required in the mean value theorem.

The corresponding point on the curve is $\left(\frac{1}{2}, \frac{1}{4}\right).$

EXERCISE 4.7

Verify the conditions of mean value theorem in the following examples 1 through 8 . In each case find a point c in the interval as stated by the mean value theorem.

1. x^2-1 on $[2, 3]$
2. x^3-2x^2-x+3 on $[0, 1]$
3. $\sin x-\sin 2x$ on $[0, \pi]$
4. $\log x$ on $[1, 2]$
5. ax^2+bx^2+cx+d on $[0, 1]$
6. ax^3+bx^2+cx+c on $[0, 1]$
7. x on $[a, b]$
8. The constant function α on (a, b)
9. Find a point on the parabola $y=(x-3)^2$,where the tangent is parallel to the chord joining (3, 0) and (4, 1)
10. Find a point on the graph of $y=x^3$,where the tangent is parallel to the chord joining (1, 1) and (3, 27).

4.8 Tangents and Normals

In this section we shall use differentiation to find the equation to the tangent of a given curve at a given point, and similarly of the normal of a given curve at a given point.

Theorem 4.11

Let $y=f(x)$ be a curve, and let (x_0, y_0) be a point on the curve. The equation of the tangent at (x_0, y_0) is

$$y-y_0=f'(x_0)(x-x_0).$$

The equation of the normal at (x_0, y_0) is

$$(y-y_0)f'(x_0)+(x-x_0)=0.$$

Proof:

Let l be the tangent at (x_0, y_0) to the curve $y=f(x)$.

Then as discussed in chapter 3, the slope of l is $f'(x_0)$, also l passes through (x_0, y_0). Therefore its equation is

$$y-y_0=f'(x_0)(x-x_0)$$

The slope of the normal at (x_0, y_0) is $-\dfrac{1}{f'(x_0)}$ if $f'(x_0)\neq 0$.

Hence, the equation of the normal at (x_0, y_0) when $f'(x_0)\neq 0$ is

$$(y-y_0)=-\frac{1}{f'(x_0)}(x-x_0), f'(x_0)\neq 0$$

This may be rewritten as

$$(y-y_0)f'(x_0)+(x-x_0)=0$$

In case $f'(x_0)=0$, the tangent is parallel to the x-axis, therefore, the normal is parallel to the y-axis. But it passes through (x_0, y_0). Therefore its equation is $x=x_0$. We note that this equation $x=x_0$ can also be written as

$$(y-y_0)f'(x_0)+(x-x_0)=0 \text{ since } f'(x_0)=0$$

Thus, the equation of the normal at (x_0, y_0) can be written as

$$(y-y_0)f'(x_0)+(x-x_0)=0$$

Example 4 26

Find the point on the curve

$y=x^3-11x+5$ at which the tangent has the equation $y=x-11$.

Solution

At a general point (x, y) on the curve, the slope of the tangent is given by

$$\frac{dy}{dx}=3x^2-11.$$

If the equation of the tangent is $y=x$, then its slope is 1.

This happens at a point (x, y) where $3x^2 - 11 = 1$.

This gives $3x^2 = 12$ and therefore $x = \pm 2$.

When $x = 2$, $y = 8 - 22 + 5 = -9$

When $x = -2$, $y = -8 + 22 + 5 = 19$

Then two points are, therefore, $(2, -9)$ and $(-2, 19)$. Of these two $(2, -9)$ lies on the line $y = x - 11$, but $(-2, 19)$ does not. Therefore, the required point is $(2, -9)$.

Example 4.27

Find the equation of the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } (x_1, y_1).$$

Solution

First we find $\frac{dy}{dx}$.

Differentiating $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to x,

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0.$$

$$\therefore \frac{dy}{dx} = -\frac{2x}{a^2} \cdot \frac{b^2}{2y} = -\frac{b^2x}{a^2y}.$$

The slope of the tangent at (x_1, y_1) is, therefore, $-\frac{b^2x_1}{a^2y_1}$.

Also, the tangent passes through (x_1, y_1).

Therefore, its equation is

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1).$$

This can be written as

$$a^2y_1(y - y_1) + b^2x_1(x - x_1) = 0.$$

This can also be written as

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}.$$

But since (x_1, y_1) lies on the ellipse, we have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

Therefore, the equation to the tangent at (x_1, y_1) becomes $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

EXERCISE 4.8

In problems 1 through 8 find the equation of the tangent and normal to the given curves at the points given .—

1. $y = x^4 - bx^3 + 13x^2 - 10x + 5$ at $(0, 5)$
2. $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $(1, 3)$
3. $y = x^3$ at $(1, 1)$
4. $y = x^3$ at $(2, 8)$
5. $y = x^2$ at $(0, 0)$
6. $x = \cos t$, $y = \sin t$, at $t = \dfrac{\pi}{4}$
7. $16x^2 + 9y^2 = 144$ at (x_1, y_1) where $x_1 = 2$ and $y_1 > 0$.
8. $y^2 = \dfrac{x^2}{4-x}$ at $(2, -2)$
9. For the curve $y = 4x^3 - 2x^5$ find all the points at which the tangent passes through the origin
10. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.
11. Show that the equation of the tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at x_0, y_0 is $\dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} = 1$.
12. Find the equation of the tangent to the parabola $y^2 = 4\,ax$ at $(at^2, 2at)$.

4.9 Differentials and Approximations

In this section, we give a meaning to the symbols dx and dy in such a way that the original meaning of the symbol $\dfrac{dy}{dx}$ coincides with the quotient when dy is divided by dx. We use this to find approximate values of certain quantities.

Let $y = f(x)$ be a given function.

Then Δx denotes a small increment in x.

We use dx also to denote the same,

We take dy as the product of dx and the already familiar derivative $\frac{dy}{dx}$ i.e. we take $dy = \left(\frac{dy}{dx}\right) . dx = f'(x)\ dx$.

On the other hand the quantity $f(x+\Delta x) - f(x)$, which is really the corresponding increment in y, is denoted by Δy.

Note that Δy and dy are not usually the same. We may regard dy as an approximate value of Δy. You can understand the geometrical meaning of Δx, Δy, dx, dy from Fig. 4.22.

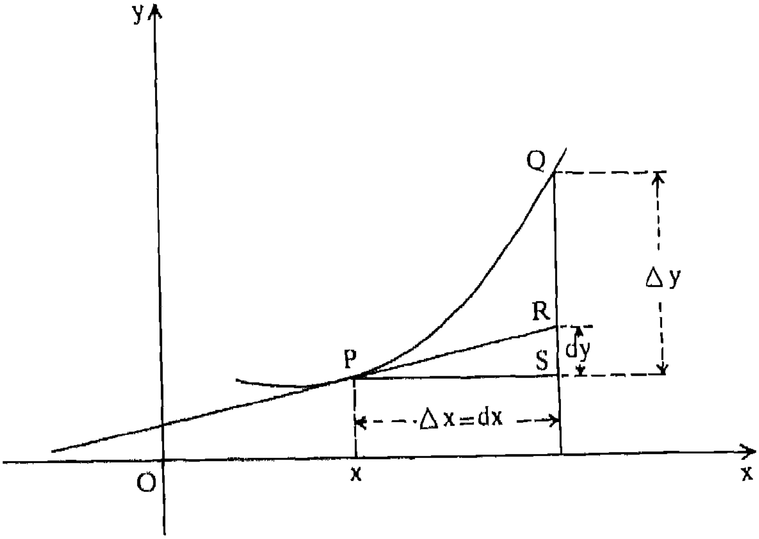


Fig 4.22

In many practical situations, it is easier to calculate dy but not Δy. dx is called the differential of x and dy is called the differential of y.

Example 4 28

Use differential to approximate $\sqrt{25.2}$

Solution

Take $y = \sqrt{x}$

$x = 25$

$dx = 0.2$

Then $\Delta y = \sqrt{25.2} - \sqrt{25} = \sqrt{25.2} - 5$

$\therefore \sqrt{25.2} = 5 + \Delta y$.

Now Δy is approximately equal to

$$dy = \left(\frac{dy}{dx}\right) dx = \frac{1}{2\sqrt{x}} \cdot (0.2)$$

$$= \frac{1}{2\sqrt{25}} (0.2) = 0.02 \cdot$$

Thus $\sqrt{25.2}$ is approximately equal to 5.02.

Example 4.29

Use differentials to approximate the cube root of 26

Solution

Let $y = x^{\frac{1}{3}}$
Take $x = 27$
$dx = -1$
So that $x + dx = 26$

Then we want $(x + dx)^{\frac{1}{3}}$. Now, $\Delta y = (x + \Delta x)^{\frac{1}{3}} - x^{\frac{1}{3}}$.

$$= (27 - 1)^{\frac{1}{3}} - 3 = (26)^{\frac{1}{3}} - 27^{\frac{1}{3}} = (26)^{\frac{1}{3}} - 3.$$

$$\therefore 26^{\frac{1}{3}} = 3 + \Delta y.$$

Here, Δy is approximately equal to dy and

$$dy = \left(\frac{dy}{dx}\right) dx = \left(\frac{1}{3} x^{-\frac{2}{3}}\right) dx$$

$$= \frac{1}{3} (27)^{-\frac{2}{3}} (-1) \text{ at } x = 27$$

$$= \frac{1}{3} \cdot \frac{1}{3^2} (-1) = -\frac{1}{27}.$$

$\therefore 26^{\frac{1}{3}}$ is approximately equal to $3 - \frac{1}{27} = \frac{80}{27}$.

EXERCISE 4.9

In problems 1 through 5, find the approximate values using differentials.

1. Cube root of 0.009
2. Fourth root of 15
3. Fourth root of 255

4. $\sqrt{401}$

5. $\sqrt{0.037}$

6. If $y = x^4 - 10$ and if x changes from 2 to 1.99, what is the approximate change in y ?

7. If $y = \sin x$ and x changes from $\frac{\pi}{2}$ to $\frac{22}{14}$ what is the approximate change in y ?

8. A circular metal plate expands under heating so that its radius increases by 2%. Find the approximate increase in the area of the plate if the radius of the plate before heating is 10 cm.

4.10 Curve Sketching

In this section, we use the results of differential calculus to sketch some curves. These results will be used to find :

In which intervals is the curve increasing ?
In which intervals is it decreasing ?
At which points does the curve take a turn ?

We can use these, together with the observations of symmetries, to sketch the curve to a considerable extent.

Example 4.30

Sketch the curve $y = x^3 - 4x$
This is a polynomial function and hence continuous on R.

Solution

Put $x = 0$. Then $y = 0$. So $(0, 0)$ is a point on the curve.

If $y = 0$, then $x(x^2 - 4) = 0$. Therefore $(2, 0)$ and $(-2, 0)$ are also points on the curve. These are the three points where the curve meets the x-axis.

Differentiating, $\dfrac{dy}{dx} = 3x^2 - 4$.

This derivative is positive when $x^2 > \dfrac{4}{3}$

i e. when either $x > \dfrac{2}{\sqrt{3}}$ or $x > \dfrac{-2}{\sqrt{3}}$

The derivative is negative in the interval $\left(-\dfrac{2}{\sqrt{3}}, \dfrac{2}{\sqrt{3}} \right)$.

Therefore, the curve is increasing upto $\frac{-2}{\sqrt{3}}$, decreasing upto $\frac{2}{\sqrt{3}}$ and then again increasing. $x = \frac{-2}{\sqrt{3}}$, is a point of local maximum and $x = \frac{2}{\sqrt{3}}$, is a point of local minimum. When $x = \frac{2}{\sqrt{3}}$, $y = \frac{8}{3\sqrt{3}} - \frac{8}{\sqrt{3}}$

$= \frac{-16}{3\sqrt{3}}$. When $x = \frac{-2}{3}$, $y = \frac{16}{3\sqrt{3}}$

Next, whenever (x, y) is a point on the curve, $(-x, -y)$ is also a point on the curve. We say that this function is an odd function.

The curve has a symmetry about the origin.

With these ideas, we sketch the curve as shown in Fig. 4.23.

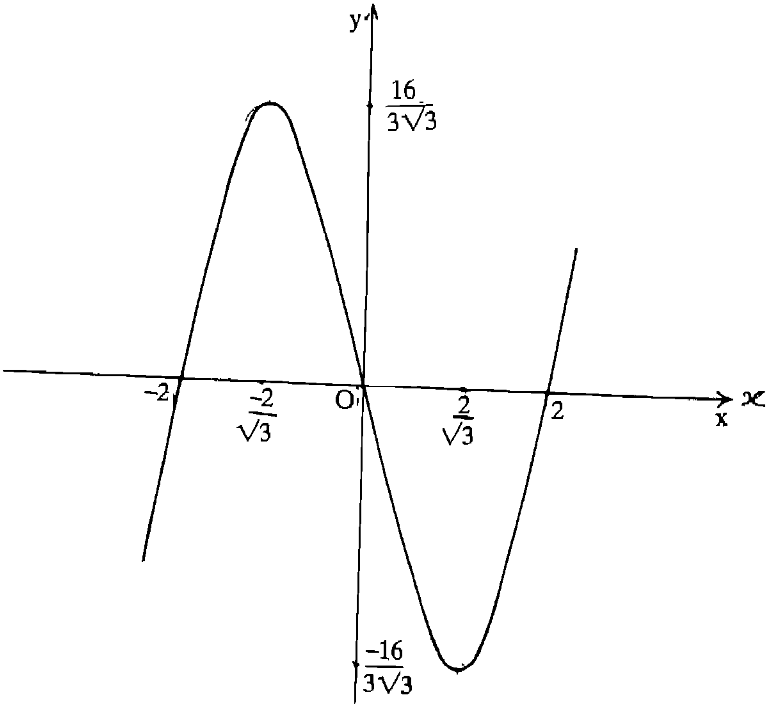


Fig. 4 23

Example 4.31

Sketch the curve $y = \sin 2x$.

Solution

First we find the points where the curve $y = \sin 2x$ meets the y-axis. For this we put $x = 0$. We get $y = 0$.

Therefore, the origin (0, 0) is on the curve

Next we find the points where the curve meets the x-axis.
For this we solve the equation $\sin 2x = 0$ We get $x = 0, \pm\frac{\pi}{2}, \pm\pi, \ldots$ or $x = \frac{n\pi}{2}$ where n is an integer.

Next we observe that $\sin 2x$ is an odd function since $\sin 2x = -\sin(-2x)$, i.e. whenever (x, y) is a point on the curve, $(-x, -y)$ is also a point on the curve. Therefore, the curve is symmetric with respect to the origin.

$$y = \sin 2x$$

$$\therefore \quad \frac{dy}{dx} = 2\cos 2x$$

$$\frac{d^2y}{dx^2} = -4\sin 2x$$

$$\frac{dy}{dx} = 0 \text{ gives } x = \pm\frac{\pi}{4}, \ \pm\frac{3\pi}{4}, \ \pm\frac{5\pi}{4}, \ \pm\frac{7\pi}{4}, \ \pm\frac{9\pi}{4}, \ldots$$

Examining the sign of $\frac{d^2y}{dx^2}$ at these points, we get that $\frac{d^2y}{dx^2}$ is negative at $x = \frac{\pi}{4}$, $\frac{5\pi}{4}$, $\frac{9\pi}{4}$, ... and at $x = \frac{-3\pi}{4}, \frac{-7\pi}{4}, \ldots$
Also $\frac{d^2y}{dx^2}$ is positive at $x = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$ and at $x = \frac{-\pi}{4}, \frac{-5\pi}{4}\ldots$ So the points $x = \frac{\pi}{4}$ $\frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$ and $x = \frac{-3\pi}{4}, \frac{-7\pi}{4}, \ldots$ are points of local maximum and the local maximum value at these points is 1. Similarly the points $x = \frac{-\pi}{4}, \frac{-5\pi}{4}, \ldots$ and
$x = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$

are points of local minimum and the local minimum value at these points is -1. Lastly we observe that $\sin 2x = \sin(2x + 2\pi) = \sin 2(x + \pi)$ for all x. This means that the periodicity of the function is π or in other words, the pattern of the curve repeats at intervals of length π. With these ideas, we sketch the curve $y = \sin 2x$ in Fig. 4.24.

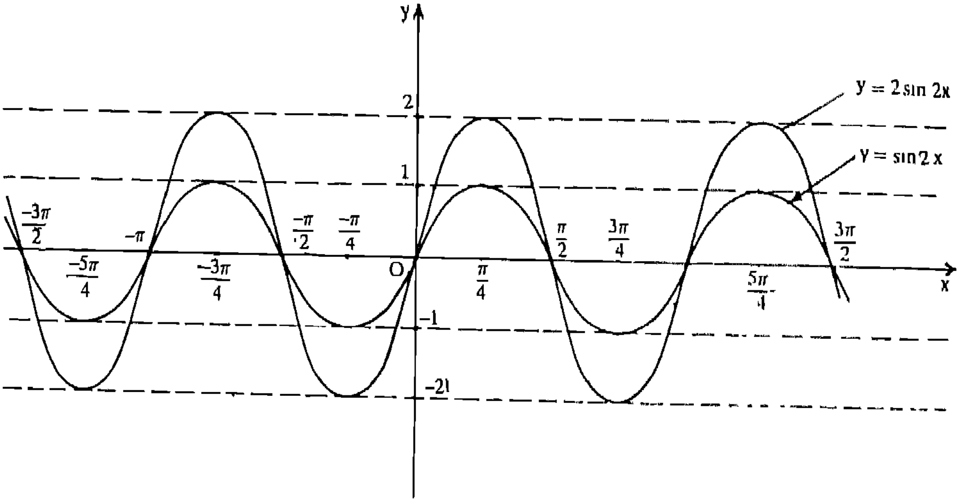


Fig 4 24

Example 4 32

Sketch the curve $y = 2 \sin 2x$

Solution : We find that the points where $y = 2 \sin 2x$ meets the coordinates axes are the same points where $y = \sin 2x$ meets them. Further the f' $(2 \sin 2x) = 0$ gives the same points as given by $f'(\sin 2x) = 0$ and the sign of $f''(2 \sin 2x)$ will be the same as the sign of $f''(\sin 2x)$ for any value of x.

This means the points of local minima or maxima of $y = 2 \sin 2x$ are the same as the points of local minima or maxima of $y = \sin 2x$, but the maximum value will be 2 and minimum value will be -2. The curve $y = 2 \sin 2x$ is sketched in Fig. 4.24 along with the curve $y = \sin 2x$.

Example 4.33

Sketch the curve $y = \sin^2 x$.

Solution . You are familiar with the graph of $y = \sin x$. Its periodicity is 2π and its maximum and minimum values are 1 and -1 respectively.

Let us now consider $y = \sin^2 x$.

First we note that it meets the x–axis in the same points where $y = \sin x$ meets the x-axis.

Again $\sin^2 x \geq 0$ for all x.

Further the maximum value of $\sin^2 x$ is 1 and the minimum value is 0. With these ideas we sketch the curve $y = \sin^2 x$ in Fig. 4.25.

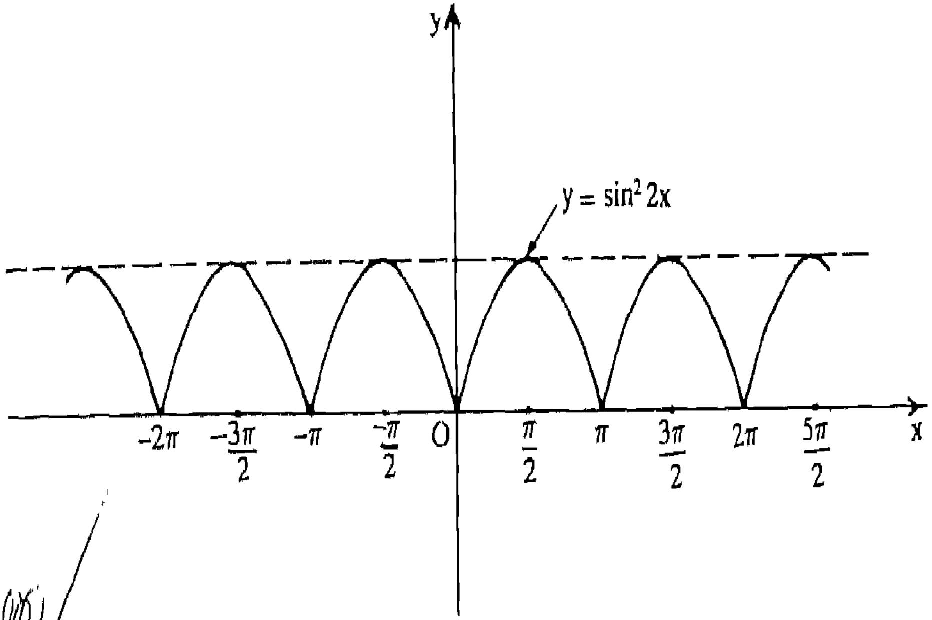


Fig. 4.25

EXERCISE 4.10

Sketch the following curves :

1. $y = 2 \cos x$
2. $y = -\sin 2x$
3. $y = x^3 + 1$
4. $y = \sqrt{9 - x^2}$
5 $y = x^2 - 1$
6. $y = (x - 1)(x - 2)(x - 3)$
7. $y = x^4 - 1$
8. $y = \sin^3 x$

CHAPTER 5

MATHEMATICAL LOGIC

5.1 What is Logic ?

The dictionary tells us that logic is the science of reasoning. It is a process by which we arrive at a conclusion from known statements or assertions with the use of valid assumptions which are known as laws of logic. In this description a number of terms have been used without proper definition. For instance, the terms, 'statements', 'valid' are taken to be common place. Logic has a role to play in any study involving reasoning. In our traditional system of learning Nyaya or Tarka Sastra (Science of Logic) was an important area of study of philosophy and it was used by philosphers in debates. In the West the Greek philosophers Plato and Aristotle were well-known logicians A mathematical treatment of logic going along with a similar treatment of set theory (known as axiomatic set theory) is now available But we shall adopt a naive approach in this text The axiomatic approach to logic was first propounded by George Boole an Englishman On this account logic relevant to mathematics is sometimes called Boolean logic. It is also called mathematical logic or more recently symbolism. With the advent of the computer and the study of language in the context of the computer, symbolic logic has gained added importance.

5.2 Basic Assumptions

In the context of logic a *proposition* or *statement* is a sentence in the grammatical sense conveying a situation which is *neither* imperative, interrogative *nor* exclamatory. In this sense, the sentence

'The cat Pussy is black in colour' is a statment, while the following sentences are *not*
'Bring Pussy the black cat here'
'Is Pussy a black cat ?'
'How Black is the cat Pussy !'.

Any statement is assumed to be either *true* or *false*, or equivalently either *valid* or *invalid* A statement cannot be both true and false at the same time. This fact is known as the *law of the excluded middle* *

'True' or 'false' is said to be truth-value of any statement according as the statement is true

---

* Some authors distinguish between a *statement* and a *proposition* as follows : A statement is a sensible combination of words while a proposition is a statement which can be definitely said to be true or false. Also, sentences such as "This cat is black in colour", "Rama will go to Madras" are taken as statements in some standard works on logic.

or false. For example, the statement

'The sum of 2 and 3 is 5'

is true and has truth value 'True' or 'T' in abbreviated notation On the other hand the statement

'Every set is a finite set'

is false and has therefore truth value 'False' or 'F'

Consider the sentence : $x + 5 = 7$. The truth of the sentence is open till we are told what x stands for. Sentence such as $x + 5 = 7$ are therefore, called '*open* sentences'. An open sentence is, thus, not a statement.

Consider the sentence

$$x + 0 = x$$

for x in a suitable domain of numbers. You know that this is a true sentence whatever be x, and so is a statement with truth value T. Identities thus turn out to be statements.

Example 5 1

Which of the following is a statement (or proposition) ?

(i) Listen to me, Krishna !
(ii) 17 is a prime.
(iii) $x^2 + 5x + 6 = 0$.
(iv) 6 has three prime factors.

Solution

(i) is imperative and is *not* proposition or a statement.
(ii) is a statement and has truth value T.
(iii) is not a statement.
(iv) is a statement and has truth value T

Example 5.2

State the truth values of the following :

(i) There are only finite number of rational numbers.
(ii) There is only one triangle apart from the triangles (congruent to it) with prescribed lengths for sides a, b, c with $a < b + c$.
(iii) The quadratic equation $ax^2 + bx + c = 0$, $a \neq o$, has always two real roots.

Solution

Truth value of (i) is clearly F, (ii) has truth value T, (iii) has truth value F, since it will not have real roots if $b^2 - 4ac < 0$.

EXERCISE 5.1

1. Find out which are statements and which are not Justify your answers
(a) Two non empty sets have always a non-empty intersection.

(b) The real number x is less than 2
(c) Two individuals are always related.

2 Write down the truth value (T/F) of the following statements :

(i) A triangle one of whose vertices lies on a circle and whose side opposite to this vertex is a diameter of the circle is a right angled triangle.
(ii) There is always a real root for any quadratic equation.
(iii) The number of ways of seating 2 persons in two chairs out of n persons is $n P_2$

5.3 Use of Venn Diagrams in Logic

Consider the statements 'All teachers are scholars'. (1)

Which is assumed to be True or equivalently having truth value T.

Does the following statement (2) follow from (1) ?

'There are some scholars who are not teachers. (2)

In other words, is the statement (2) also true ?

Venn diagrams may be used to help us answer such questions. Let U represent the Universal set of human being and let T represent the set of all teachers and let S represent the set of all scholars. Consider the Venn diagram in Fig. 5.1.

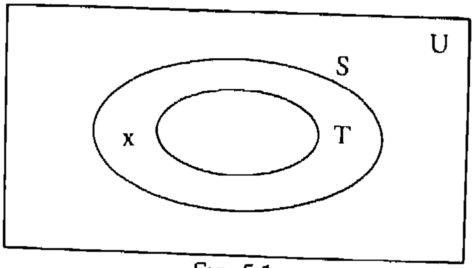


Fig. 5.1

Does this Venn diagram represent the truth of th statement (1) ? Yes, it represents. Does it also represent the truth of the statement (2) ? Yes, the diagram shows that there is a scholar x who is not a teacher.

Again consider the Venn diagram in Fig. 5 2

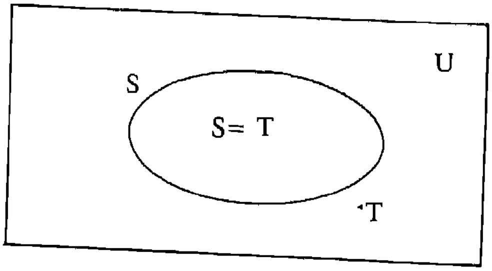


Fig. 5 2

Does this Venn diagram represent the truth of the statement (1) ? Yes, it does.

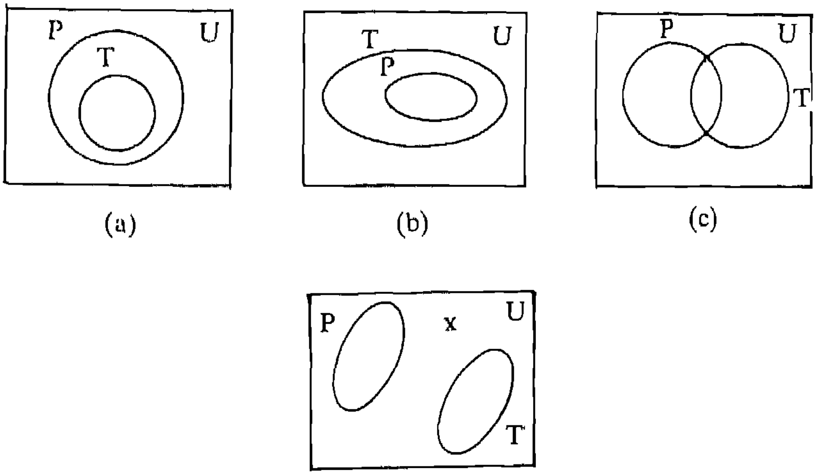
Does it represent the truth of the statement (2) also ? No, since any element of S is in T. Thus you see that the statement (2) does not follow (logically) from statement (1) or we may also say that statements (1) and (2) are not logically equivalent

Consider the statement :
'No policeman is a thief'. (3)
Which is assumed to be true
Also, consider the following statements :
'Thieves are not policemen'. (4)
'Men who are not policemen are thieves'. (5)

Do statements (4) and (5) follow from statement (3) or, in other words, are (4) and (5) also true ?

Let us make use of Venn diagrams again. Let the Universal set U be the set of human beings.

Let P represent the set of all policemen and let T represent the set of all thieves. Which of the following Venn diagrams represent the truths of the statement (3) ?



(a) (b) (c)

(d)

Fig. 5.3

You agree that (d) in Fig. 5.3 represents the truth of the statement (3).

Does (d) represent the truth of the statement (4) ? Yes.

Does (d) also represent the truth of the statement (5) ? No. For example 'x' is not a policemen and he is not a thief also. So (5) is false.

Thus, statements (3) and (4) are logically equivalent.
However, statements (3) and (5) are not logically equivalent.

Consider the statement :

'Some teachers are scholars' . (6)

As you can see the Venn diagram in Fig 5.4. represents the truth of (6).

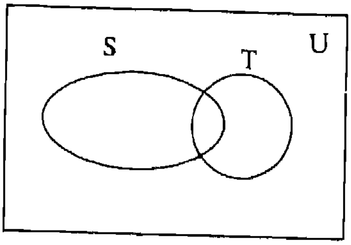


Fig. 5.4

Consider the following statements in this context :

'There are teachers who are scholars' . (7)

'There are teachers who are not scholars' . (8)

'There are scholars who are teachers' (9)

'There are scholars who are not teachers' (10)

'There are persons who are teachers and scholars' . (11)

Which of the statement from (7) to (11) are logically equivalent to (6) ?

Refer to the Venn diagram in Fig. 5.2 again. It represents the truth of the statement

'All teachers are scholars and all scholars are teachers' . (12)

also expressed as

'Teachers are precisely scholars' . (13)

(12) and (13) are said to be equal or one and the same

Example 5 3.

Represent the truth of the statement ·

'All rational numbers are real numbers' by means of a Venn diagram.

Solution

With the usual notation for rational and real numbers, we have the diagram given in Fig. 5.5.

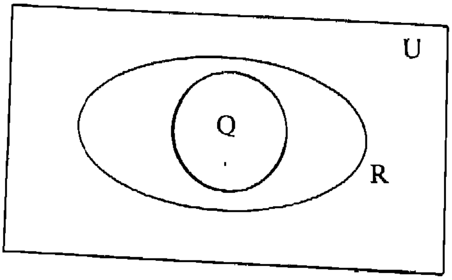


Fig. 5.5

Example 5.4.

Give the Venn diagram for the truth of the following statement :

'Equilateral triangles are isosceles triangles' .

Solution

Denoting the set of equilateral triangles by E and that of isosceles triangles by S, we have the diagram in Fig. 5.6.

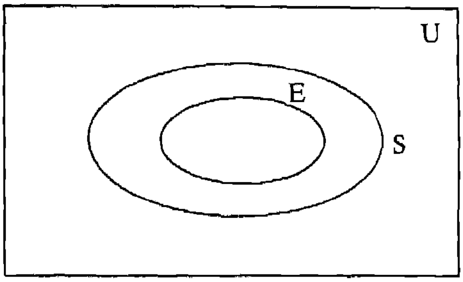


Fig. 5.6

EXERCISE 5.2

1. Assuming the truth of the statement 'All equilateral triangles are equiangular and all equiangular triangles are equilateral' represent it in a Venn diagram.
2. 'Some quadratic equations have two real roots' Express the truth of the above statement by means of a Venn diagram.

5.4 Basic Logical Connectives or Logical Operators

Any statement or proposition whose truth or otherwise does not explicitly depend on another statement is said to be *simple.* For instance,

'Madras is the capital of Kerala',
'The set of real numbers is infinite'

are simple statements.

In other words, a simple statement is *not* in any way a combination of two statements. On the other hand, a *compound statement*, is a combination of two or more simple statements.

For instance,

(i) $\sqrt{3}$ is a real number and $\frac{1}{2}$ is a rational number

(ii) The school works or a holiday is declared

(iii) If it rains, then the school may be closed

(iv) If a and b are rational numbers, then their product is not real

(v) The diagonals of the quadrilateral are perpendicular if quadrilateral is a rhombus

(vi) It is not true that $\sqrt{3}$ is rational

are compound statements. These statements are obtained by combining or connecting two simple statements. Take, for example, (i). It is obtained by connecting, the statements

'$\sqrt{3}$ is a real number'

and

'$\frac{1}{2}$ is a rational number',

using the conjunction 'and', (iii) is obtained by connecting the statements

'It rains'

and 'The school may be closed',

using the phrase 'If.........then'.

It is important to note that when defining a compound sentence we are *not* bothered about its truth or otherwise. For example; $3 > 1$ and $5 < 2$ is a compound statement. The phrases or words which connect two simple statements are called *sentential connectives, logical connectives or simply connectives.*

The simple statements combining which a compound statement arises are called the constituents or *components* of the compound statement. The possible connectives are as follows :

	Connective	*Compound statement formed by the connective*	*Symbol used for the connective*
(α)	and	conjunction	∧
(β)	or	disjunction	∨
(γ)	if... then	implication (conditional)	(⇔)* →
(δ)	if and only if (iff)	(biconditional), equivalence	(⇔)* ↔
(∊)	not	negation	(¬)* ~

(α) *Conjuntion*

Two simple statements connected by the word 'and' are said to form a conjunction.

For example,

'Rama is a boy and Sita is a girl'

'$8 > 5$ and $7 < 5$'

* The alternative symbolism in brackets is also in use. How exactly the symbols are placed will be clear from what follows :

are conjunctions.

It is useful to have some notation to represent statements. Let us represent the statement by lower case letters like $p, r, s, \ldots$. Thus we write

$p : 8 > 5$ and
$q : 3 < 5$.

The conjunction formed by p and q is denoted by $p \wedge q$.

How is the truth value of a conjunction related to the truth values of the components ?

(1) *If p, q are two statements, p has truth value T, q has truth value T then we take $p \wedge q$ too to have truth value T.*

Let us consider some illustration to see how this is natural.

Illustration

Let $p : 8 > 5$ and
$q : 3 < 5$.
$\therefore p \wedge q . 8 > 5$ and $3 < 5$.
What is the truth value of $p \wedge q$ as it is ?
The truth value of p is T and of q is also T.
We not that $p \wedge q$ has also truth value T.
(2) *If p has truth value T and q has truth value F we take $p \wedge q$ to have truth value F.*

Illustration

$p : 8 > 5$ (T).
$q \cdot 3 < 1$ (F).

The truth values are as indicated against p and q.
Now $p \wedge q : 8 > 5$ and $3 < 1$.
The truth value of $p \wedge q$ is clearly F.

(3) *If both p and q have truth value F, we take $p \wedge q$ to have truth value F.*

Illustration

$p : 8 > 11$ (F)
$q : 5 < 3$ (F)
with truth values as indicated.
Then $p \wedge q : 8 > 11$ and $5 < 3$ has truth value F, as you can easily see.

We can summarise the above discussion in the form of the following table :

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(β) *Disjunction or Alternation*

Two simple statements connected by the word 'or' are said to form a disjunction. This amounts to an alternation of two statements.
For example, the compound statements
' The sun shines or it rains ',
'$2 + 3 = 5$ or $3 > 5$',

'$2 + 3i$ is a real number or it is a complex number' are disjunctions.

The disjunction formed by the simple statements p and q is denoted by $p \vee q$.

As with conjunction let us develop the truth table of disjunction.

(1) If *p, q are two statements which have truth value T, then $p \vee q$ is taken to have truth value T.*

Illustration

Let, for instance $p : 8 > 3$ (T)
$q : 5 > 2 + 1$ (T)
the truth values being as indicated.

So $p \vee q : 8 > 3$ or $5 > 2 + 1$.
The truth value of $p \vee q$ is T as it should naturally be.

(2) If *p has truth value T and q has truth value F, then $p \vee q$ is taken to have truth value T.*

Illustration

To illustrate this, let
$p : 8 > 3$ (T) and
$q : 5 < 2$ (F)

with the truth values as indicated.
So, $p \vee q : 8 > 3$ or $5 < 2$.

The truth value of $p \vee q$ is clearly T.

(3) If *p and q have both truth value F, $p \vee q$ too is taken to have truth value F.*

For example; if $p . 8 > 10$ (F) and
$q . 5 < 3$ (F)
with the truth values as indicated, then
$p \vee q : 8 > 10$ or $5 < 3$.

The above statement is obviosuly false, i.e. it has truth value F

Now summarising the above discussion, we have the following truth table :

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Remark

We do not bother about the order in which the component statements of a conjunction or disjunction occur. This would be clear from the truth tables as well.

In symbols $p \wedge q = q \wedge p$, $p \vee q = q \vee p$.

In the above symbolism '=' stands for the 'same as' in commonsense language This conjunction holds in what follows. In due course, it will be shown to be replaceable by another symbol '$\leftrightarrow$' known as 'equivalence'.

(γ) *Implication or Conditional Statement*

Two statements connected by the connective phrase 'if... then' give rise to a compound statement known as implication.

For example,

'If it rains, then the atmospheric humidity increases',
'If a is a rational number, then $\frac{a}{2}$ is an irrational number',
'If ABC is a triangle, then $\angle A + \angle B + \angle C$ is equal to two right angles'
are implications.

If p, q are two simple statements, forming the implication
'If p then q',
then we denote this implication by $p \rightarrow q$ in symbols.

Let us develop the truth table of 'implication' or conditional statements.

(1) *If both p and q are true we take that $p \rightarrow q$ is also true.*

Illustration

Let p : The number 14232 ($=N$) is divisible by 3 (T) and
q : The sum of the digits forming N is divisible by 3 (T).

In this case
$p \rightarrow q$: If the number N is divisible by 3, then the sum of the digits forming N is divisible by 3.
and its truth value is T.

(2) *If p is true and q is false, $p \rightarrow q$ is taken to be false.*

Illustration

Let p : The number 14232 ($=N$) is divisible by 3	(T)
q : The sum of the digits forming N is not divisible by 3	(F)

Then $p \rightarrow q$: If the number N is divisible by 3, then the sum of the digits forming N is not divisible by 3.

As you know very well, the truth value of $p \rightarrow q$ is F.

(3) *If p is false and q is true, $p \rightarrow q$ is taken to be true.*

Illustration

Let p : The number 14231 (=N) is divisible by 3 (F)
q : The sum of the digits forming N (i.e. $1 + 4 + 2 + 3 + 1$) is not divisible by 3 (T)

Here $p \rightarrow q$: If the number 14231 is divisible by 3, the sum of the digits $(1 + 4 + 2 + 3 + 1)$ is not divisible by 3.

is taken to be true though commonsense cannot help us to infer so.

This assumption is made to be consistent with the assumptions.

(4) *If both p and q are false we take that $p \rightarrow q$ is true.*

Illustration

Let p : The number 14231 (=N) is divisible by 3 (F)
q : The sum of the digits forms N (i.e. $1 + 4 + 2 + 3 + 1$) is divisble by 3. (F)

So $p \rightarrow q$: If the number 14231 is divisible by 3, then $1 + 4 + 2 + 3 + 1$ is divisible by 3 has truth value T since p and q have truth value F.

The following is the truth table of $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(δ) *Equivalence or Biconditional Statement*

In your study of mathematics, you have come across statements which involve the phrase 'if and only if'. For example you read the following :

A number is divisible by 3 *if and only if the sum of the digits forming the number is divisible by 3.* (1)

What does this statement really mean? This means :

If the sum of the digits forming the number is divisible by 3, then the number is divisible by 3 (2)

which is an implication and (conversely)

If the number is divisible by 3, then the sum of the digits forming the number is divisible by 3 (3)

which is also an implication.

Thus the statement (1) is the conjunction of two conditional statments (implications). Therefore, the statement (1) is called a 'biconditional statement' or 'equivalence'.

If p, q are two simple statements, then the compound statement :

$$p \rightarrow q \text{ and } q \rightarrow p$$

is called a biconditional statement formed by p and q and is denoted by

$$p \leftrightarrow q.$$

You know

'Two triangles ABC and DEF are congruent if and only if their corresponding sides are equal'.

This means : ABC, DEF are congruent triangles (p) → the corresponding sides of Δ's ABC and DEF are equal (q)

and

The corresponding sides of Δ's ABC
and DEF are equal (q) → ABC and DEF are congruent triangles (p).

The following are also biconditional statements :

'A triangle is equilateral if and only if it is equiangular'.
'l is perpendicular to m if and only if m is perpendicular to l'.

We have already seen that $p \leftrightarrow q = (p \to q) \wedge (q \to p)$. Therefore,we can easily construct the truth table of $p \leftrightarrow q$, with the help of the truth table for a conjunction.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q = (p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note that the above truth table incorporates the truth tables for $p \to q$ and $q \to p$ too.

(e) *Negation*

An assertion that a statement fails or a denial of a statement is called the negation of the statement.

In our daily usage, we make statements like :

'Rama is good.'

What should be the negation or denial of this ?
Obviously is

'It is not true that Rama is good'

or what is the same

'Rama is not good'.

Of course, if we are agreed that anything which is not good is necessarily bad, the above statement can also be written as

'Rama is bad'.

As a second example consider the statement

'All mathematicians are men.'

What do you think its negation should be ?
Could it be . 'All mathematicians are women' ?
Logically we do not consider 'All mathematicians are women' as the negation of 'All mathematicians are men'.

The negation of the statement 'All mathematicians are men' is taken as :

'All mathematicians are *not* men'.

Which is the same as the statement

'There exists a mathematician who is *not* a man'.

Consider the statement

'There is a complex number which is *not* a real number'.

The negation in this case is 'All complex numbers are real numbers'.

The negation of the statement

'$3 > 5$'

is the statement

'$3 \leq 5$'

which can also be written as

'$3 \not> 5$'.

If p is a statement, then its negation is denoted by $\sim p$.

The truth table of negation is as follows :

p	$\sim p$
T	F
F	T

Example 5.5

Give the truth table for the statement $\sim p \vee q$.

Solution

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Example 5.6

Write down the truth table for $\sim p \wedge \sim q$.

Solution

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

EXERCISE 5.3

1. Write down the truth table for the statement
   $(\sim p \vee q) \wedge (\sim p \wedge \sim q)$
2. Give the truth table for $l \leftrightarrow m$ where
   $l = (p \rightarrow q) \wedge (q \rightarrow p)$ and $m = p \leftrightarrow q$
3. Give a truth table for the statement
   $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$
4. Write down the truth table for the statment
   $(p \wedge q) \rightarrow \sim p$
5. Give the truth table for the statement
   $(p \wedge q) \rightarrow (p \vee q)$

5.5 Negation of Compound Statements

Writing the negation of a statement is not that simple, particularly when it is a compound statement having conjunctions, disjunctions, implication, equivalence etc. Hence let us discuss the negation of compound statements.

Negation of Conjunction

'Mohammed is in class X and Xavier is in class VII' has its negation the denial of its components namely

'Mohammed is *not* in class X'

and

'Xavier is not in class VII'

which are simultaneous statements. Thus the denial would imply either

(a) denial of any one of these statements

or (b) denial of both of these statements.

The nature of the description in (a), (b) precisely indicates that the required negation is a disjunction of the negation of the component statements, viz.

'Mohammed is not in class X *or* Xavier is not in class VII.' Note that the denial or negation is *not* the statement

'Mohammed is not in class X *and* Xavier is not in class VII'.

We have the following formula for the negation of a conjunction :

$$\sim (p \wedge q) = \sim p \vee \sim q$$

Negation of Disjunction

Consider the statement

$$a > -7 \text{ or } a < 7$$

which consists of the components

(α) $a > -7$
(β) $a < 7$.

The failure or denial of this compound statement arises only when both (α) and (β) fail simultaneously. Thus the desired negation is

$a \not> -7$ *and* $a \not< 7$

or, what is the same

$a \le -7$ and $a \ge 7$

or in a more compact form

$|a| \ge 7$.

Likewise, the statement

'Haq is cruel or he is strict'

or, what is the same,

'Haq is neither cruel nor strict.'

We have the following formula for the negation of a disjunction

$\sim (p \vee q) = \sim p \wedge \sim q$

The truth table given below supports the formula stated above for the negation of conjunction and disjunction.

p	q	$p \wedge q$	$p \vee q$	$\sim(p \wedge q)$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$\sim p \vee - q$
T	T	T	T	F	F	F	F	F	F
T	F	F	T	T	F	F	T	F	T
F	T	F	T	T	F	T	F	F	T
F	F	F	F	T	T	T	T	T	T

Negation of Implication

'Rama is tall, therefore he is slim' has the negation

'Rama is slim *not* because he is tall'

More explicitly, the negation is

'Rama's being tall does not *imply* (or mean) that he is slim.'

If we write

p : Rama is tall
q : Rama is slim,

the given statement 'Rama is tall, therefore he is slim' is written as

$p \rightarrow q$

which in words, is :

'If p, then q.' or 'p, therefore q' or 'p implies q'

The negation of '$p \rightarrow q$' is taken as :

'p and not q.'

In the verbal form the negation of the given statement is therefore

'Rama is tall and he is not slim.'

We have the followig formula for the negation of an implication

$\sim (p \rightarrow q) = p \wedge \sim q$

This is again supported by the following truth table :

p	q	$p \rightarrow q$	$\sim q$	$p \wedge \sim q$	$\sim (p \rightarrow q)$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	T	F	F

Negation of Biconditional Statement or Equivalence

Let us consider the compound statement

'India will be prosperous if and only if its citizens are industrious.'

The components of this statement are

p : India will be prosperous.
q : Indian citizens are industrious.

The compound statement is the equivalence

$p \leftrightarrow q$.

Its negation is the denial of $p \rightarrow q$ and $q \rightarrow p$ as simultaneous statements.

This clearly means that

either $p \rightarrow q$ is denied *or* $q \rightarrow p$ is denied.

In words, this means that the negation of $p \leftrightarrow q$ is the statement

'Either India will be prosperous and Indian citizens
are not industrious or India will not be prosperous

and Indian citizens are industrious'.

As another example consider the statement

'$|a| < 2$ if and only if $a < 2$ and $a > -2$.'

Its negation is

'Either $|a| < 2$ and $a \geq 2$ or $a \leq -2$
or $a < 2$ and $a > -2$ and $|a| \geq 2$.'

Note that in the above two verbal descriptions 'either. or' does not preclude the conjunction of the two relevant statements.

We see that ·

$$\begin{aligned} p \leftrightarrow q &= (p \rightarrow q) \wedge (q \rightarrow p) \\ \text{So} \sim (p \leftrightarrow q) &= \sim [\,(p \rightarrow q) \wedge (q \rightarrow p)\,] \\ &= [\,\sim (p \rightarrow q)\,] \vee [\,\sim (q \rightarrow p)\,] \\ &= [\,p \wedge \sim q\,] \vee [\,(q \wedge \sim p)\,]. \end{aligned}$$

So we have the following formula for the negation of an equivalence

$\sim (p \leftrightarrow q) = (p \wedge \sim q) \vee (q \wedge \sim p)$.

It is left as an exercise to the student to verify this formula by writing the truth table.

A word of caution is necessary about writing down the negation of a statement. Though the negation of the statement p is the statement

'p is not true'

more often there is need to rewrite the negation in a suitable form But this requires care. Take the statement,

'All differentiable functions are continuous.'

The negation is

'It is not true that all differentiable functions are continuous.'

This is *not* equivalent to

'Differentiable functions are not continuous '

while it is certainly equivalent to

'All differentiable functions are not continuous' or what is the same

'There exists a differentiable function which is not continuous.'

Incidentally note that when writing down the negation we are not bothered about the truth or validity of either the statement or its negation.

Note

It is clear that $\sim (\sim p)$ is the same as p.

Remark

As the symbol ~ applies only to a single statement it is a ***modifier*** rather than a *connective*, though it is listed so. It is a denial or a failure of a statement.

Example 5.7

If p : lines l and m are perpendicular to each other,
q : A is a point on m,
write down in symbols the statement
r : A is a point on the line m which is perpendicular to l.

What is the negation of this statement ?

Solution

Clearly r is $p \wedge q$.
We know that $\sim (p \wedge q) = \sim p \vee \sim q$.
In this case $\sim p = l$ is not perpendicular to m
and $\sim q = A$ is not a point on m
Thus $\sim (p \wedge q) = l$ is not perpendicular to m *or* A is not a point on m.

Example 5.8

If p : I study
q : I fail,
what is the symbolism for the statement
r : I study or I fail.
What is the negation of this statement r ?

Solution

Evidently r is $p \vee q$.
We know that
$\sim (p \vee q) = \sim p \wedge \sim q$
$\therefore \sim r = \sim p \wedge \sim q$ is
'I do not study and I donot fail'.

Example 5.9

Write down the statement 'A complex number is a real number' in the form of a compound statement.

Solution

If p : x is a complex number
q : x is a real number
are taken as statements,
then the desired compound statement is clearly

$$p \rightarrow q.$$

(In mathematics, this logical symbol is more often written as $\Rightarrow$)

Note

We showed that

(i) $\sim(p \wedge q) = \sim p \vee \sim q$

(ii) $\sim(q \vee q) = \sim p \wedge \sim q$.

The symbol '=' in (i) and (ii) above is used in the sense of 'is' or 'is the same as'. Clearly '=' can be replaced by $\leftrightarrow$' viz. equivalence. .

Remark

The formula $\sim(p \wedge q) = \sim p \vee \sim q$ and $\sim(p \vee q) = \sim p \wedge \sim q$ are also called De Morgan's laws since the De Morgan's laws of set theory arise by interpreting the symbols $\sim$, $\vee$, $\wedge$ as complementation, union, and intersection respectively.

Now $\sim p$: Triangle is not equilateral
$\sim q$: Triangle is not equiangular

We know $\sim(p \leftrightarrow q) = (\sim p \wedge q) \vee (p \wedge \sim q)$

$l = p \wedge \sim q$: There exists an equilateral triangle which is not equiangular.

$m = \sim p \wedge q$: There exists an equiangular triangle which is not equilateral.

So the required negation is $l \vee m$. In words this negation is :
'There exists either an equilateral triangle which is not equiangular or an equiangular triangle which is not equilateral.'

EXERCISE 5.4

1. 'Ram is smart and healthy',
   'Ram is neither smart nor healthy',
   Are these statements negations of each other ?
2. Are the following statements negations of each other ?
   x is not a rational number.'
   'x is not an irrational number.'
3. If p, q are two statements show that
   $\sim(\sim p \wedge \sim q) = p \vee q$
   and $\sim(\sim p \vee \sim q) = p \wedge q$
4. If p stands for the statement 'I like tennis' and q for the statement, 'I like foot-ball', what does $\sim p \wedge \sim q$ stand for ?

5.6 Duality

In geometry, we have what is known as *dual relationship* between 'line' and 'point' through the interchange of the words 'meet' and 'join'. For example consider the following two statements from geometry

'A *line* is the join of two *points*'
'A *point* is the *meet* of two *lines*'.

Interchange 'line' and 'point', 'join' and 'meet' in the first statement. What do you get ? You get : 'A point is the meet of two lines' which is the second statement Similarly you get the first statement from the second. With respect to statement p and *q*, we have the following

(i) $\sim (p \wedge q) = \sim p \vee \sim q$
(ii) $\sim (p \vee q) = \sim p \wedge \sim q$

As you can verify, if you replace $\wedge$ by $\vee$ in (i), you get (ii). In the same way you get (i) from (ii). (i) is said to be the dual of (ii) with respect to $\wedge$ and $\vee$. In the same way (ii) is the dual of (i).

Example 5.10

Write down the statement 'Two congruent triangles are precisely those which have corresponding sides equal' as an equivalence and write its negation also.

Solution

If p : Two triangles are congruent,
q : Two triangles have corresponding sides equal, then the given statement is the same as

$$p \leftrightarrow q$$

Using the formula given earlier, the required negation is $(\sim p \wedge q) \vee (p \wedge \sim q)$, which expressed in words is 'Two triangles are not congruent and have corresponding sides equal *or* two triangles are congruent and have a pair of corresponding sides equal'.

Example 5.11

Write down the negation of the statement 'All the sides of an equiangular triangle are of the same length.'

Solution

The negation is 'Not all the sides of an equiangular triangle are of the same length'.
It can be restated as 'There exists an equiangular triangle, two of whose sides are *not* of the same length.'

Example 5.12

Write down the negation of the statement 'A triangle is equilateral if and only if it is equiangular.'

Solution

If p : Triangle is equilateral
q : Triangle is equiangular,
then the given statement is $p \leftrightarrow q$.

5.7 Statement Patterns or Well-formed Formulas

If $p, q\ r, \ldots$ are statements (which can be specialised i.e. which can be treated as variables) then any statement involving these statements and the logical connectives $\wedge, \vee, \rightarrow, \leftrightarrow, \sim$ is called a *statement pattern* or a *well-formed formula.*

The following are statement patterns .

(i) $p \vee q$
(ii) $p \rightarrow q$
(iii) $((p \wedge q) \vee\ r) \rightarrow a \wedge \sim a$
(iv) $(p \leftrightarrow q) \wedge (r \leftrightarrow s)\ \vee (t \rightarrow s)$

Note.

Care should be exercised in introducing or removing brackets in statement patterns. For example $\sim (p \vee q)$ is different from $\sim p \vee q$ while $(p \wedge q) \wedge r$ could be written as $p \wedge q \wedge r$.

Note

A statement is also a statement pattern. Statement patterns also give rise to truth tables. If the number of variables is n, since each variable has two truth values the number of rows required in the table is 2^n, which becomes large when n is large. Thus truth tables become unwieldy if the number of variables is large. The following are examples of truth tables for statement patterns.

Example 5.13

The truth table for $l \wedge m$ where $l =: \sim q \rightarrow \sim r$, $m = \sim r \rightarrow \sim q$ is

q	r	l	m	$l \wedge m$
T	T	T	T	T
T	F	T	F	F
F	T	F	T	F
F	F	T	T	T

Example 5.14

If $l = \sim (p \vee q)$, $m = \sim p \wedge \sim q$, the truth table for $l \leftrightarrow m$ is

p	q	$p \vee q$	$\sim (p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$l \leftrightarrow m$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

1. Write down the truth table for the following statement patterns :
   (i) $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
   (ii) $(p \rightarrow q) \rightarrow [\ (q \rightarrow r) \rightarrow (p \rightarrow q)\ ]$

5.8 Tautology, Contradiction

Observe in Example 5.14 the last column of the truth table for $l \leftrightarrow m$ has all the entries as T. This means that whatever be the assumption about the variable statements, p, q, viz. whether they are true or false, $l \leftrightarrow m = \sim (p \vee q) \leftrightarrow \sim p \wedge \sim q$ is always true. Such a statement pattern is called a *tautology* or *theorem* or a *logically valid statement pattern* * It can easily be checked that the statement pattern (ii) in Exercise *5.5* is also a tautology.

The statement pattern $p \vee \sim p$ is a simple example of tautology. The truth table in this case is the following :

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Because of the observation made in § 5.7 about the size of the truth table for large values of the number of components or variables, tautologies cannot always be easily identified through truth tables. At the other extreme we have a statement pattern which has always truth value F whatever be the truth values of the variables. Such a statement pattern is said to be a *contradiction*. $(p \wedge \sim p)$ is a simple example of a contradiction. This can be seen from its truth table below :

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Example 5.15

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution

The assertion is clear from the following truth table :

p	q	$p \vee q$	$p \wedge q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
F	T	T	F	T
T	F	T	F	T
F	F	F	F	T

EXERCISE 5.6

1. Show that $(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$ is a tautology.
2. Show that $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction.
3. Show that $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$ is a tautology.
4. Show that $p \rightarrow (p \vee q)$ is a tautology.

---

*Tautologies are symbolically indicated by the symbol $\models$ preceding them. For example $\models p \vee \sim p$

5.9 Statement Patterns and Switching Circuits

A switching circuit is a connection of a finite number of switches p, q,..., the question under consideration being whether electric current will flow in the circuit for a given situation of the switches. Note that there are only two possibilities for any switch viz., it is 'on' or it is 'off' Likewise for any circuit there are only two possibilities, viz., the current either flows through or not.

To every switch or circuit are attached two values 0 and 1. If in a situation the current does not flow through the switch or circuit from one end to the other the value 0 is attached to the switch or circuit in that situation. If the current flows through the switch or from one end to another of a circuit the value 1 is attached to the switch or circuit as the case may be The value 0, 1 attached to the switches or circuits are called their transmittance or flow values. A switch can also be considered as a trivial case of a circuit.

No distinction is made between two switches which are simultaneously in the 'on' or 'off' position in any situation. The same letter is used to denote such switches Two circuits with the same switches (as described just now) in which current either flows from one end to the other or does not flow are said to be *equivalent*. Reduction of number of switches to obtain an equivalent curcuit is called *simplification* of the circuit

Note that if distinct letters are used to denote two switches the 'on' or 'off' position of these two switches is independent of each other

A switch which is always 'on' is denoted by 1. On the other hand a switch which is always 'off' is denoted by 0 These switches are equivalent respectively to a circuit in which current always flows from one end to the other and does not flow from one end to the other always irrespective of the position of the switches constituting it. On this account these circuits also are denoted respectively by 1 and 0

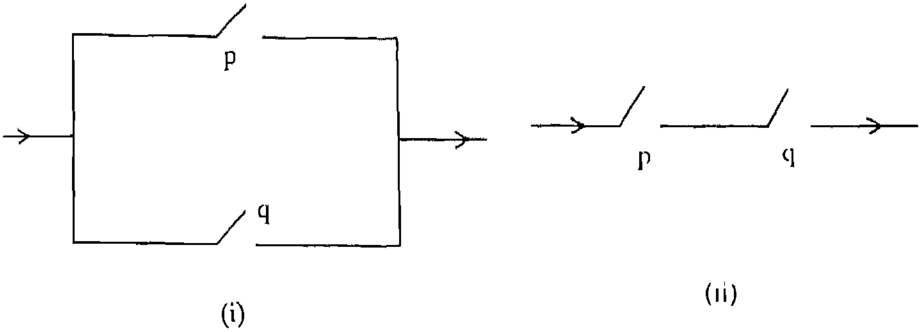


Fig 5.7

In Fig 5 7 (i) above the switches p, q are connected in parallel. The resulting circuit is denoted by $p + q$ In Fig. 5.7 (ii) the switches p, q are connected in series. The resulting circuit is denoted by $p \cdot q$

Note that current flows from one end of $p + q$ to anothe if and only if one of p or q (meaning one or both) is on and *not* otherwise. Similarly current flows from one end to the other of $p \cdot q$ if and only if both the switches p, q are on and *not* otherwise. These facts are enshrined in the following flow tables for $p + q$ and $p \cdot q$.

Flow Table for $p + q$

p (input)	q (input)	$p + q$ (output)
1	1	1
1	0	1
0	1	1
0	0	0

Flow Table for $p{\cdot}q$

p (input)	q (input)	$p{\cdot}q$ (output)
1	1	1
1	0	0
0	1	0
0	0	0

You may see that in the flow table for $p + q$, if we replace 1 by T, 0 by F and + by $\vee$, the resulting table will be the truth table for the statement pattern $p \vee q$. Similarly in the flow tabel for $p{\cdot}q$, if we replace 1 by T, 0 by F and$\cdot$ by $\wedge$, the resulting table will be the truth table for the statement pattern $p \wedge q$. Thus $p + q$ can be identified with the statement pattern $p \vee q$ and $p{\cdot}q$ can be identified with the statement pattern $p \wedge q$. Flow table can therefore be called truth tables as well. Flow tables are also called input-output tables.

As regards the operation symbols + and $\cdot$ which stand respectively for parallel and series connections, these are used for circuits as well, since such connections can be made between two circuits too. As an example, the circuit in Fig. 5.8 is $(p{\cdot}q) + r$.

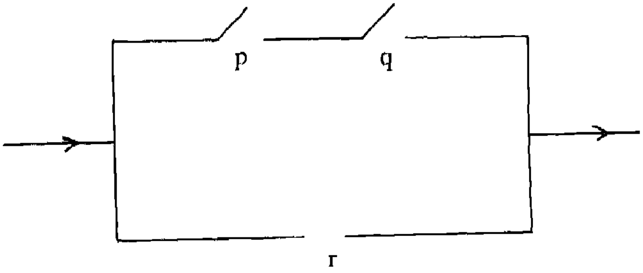


Fig 5.8

Example 5 16

Write the truth table for the switching circuit of Fig 5 8.

Solution

In symbolic representation, the circuit is $(p{\cdot}q) + r$. The truth-table is the following ·

Inputs			outputs
p	q	r	$(p{\cdot}q) + r$
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	0

Example 5.17

Write the truth table for the circuit shown in Fig. 5.9 and show how it can be replaced by a simpler circuit.

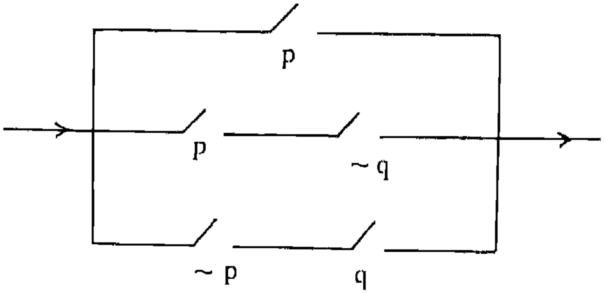


Fig. 5.9

Solution

In symbolic form the given circuit is $p + (p\cdot \sim q) + (\sim p\cdot q)$

The following is the truth table for the circuit :

p	q	$\sim p$	$\sim q$	$p\cdot \sim q$	$\sim p\cdot q$	$p + (p\cdot \sim q) + (\sim p\cdot q)$
1	1	0	0	0	0	1
1	0	0	1	1	0	1
0	1	1	0	0	1	1
0	0	1	1	0	0	0

Compare this truth table with the truth table for $p + q$ given below :

p	q	$p + q$
1	1	1
1	0	1
0	1	1
0	0	0

You find that $p + (p\cdot \sim q) + (\sim p\cdot q) = p + q$.

Hence, the given circuit can be simplified as shown in Fig. 5.10.

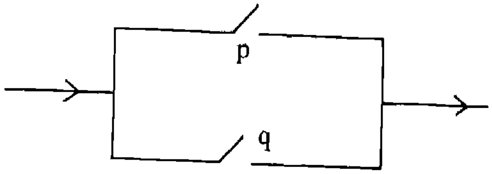


Fig. 5.10

EXERCISE 5.7

1. Write down the truth table for the switching circuit of Fig. 5.11.

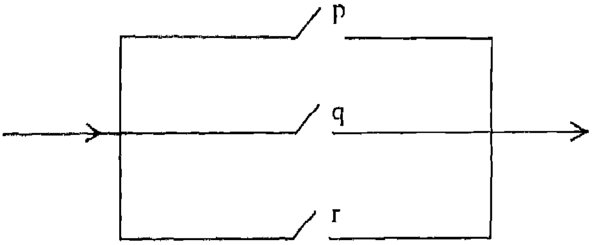


Fig. 5.11

2. Write the truth table for the circuit of Fig. 5.12.

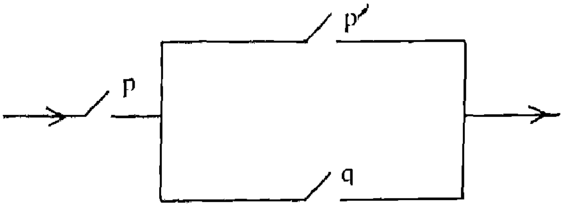
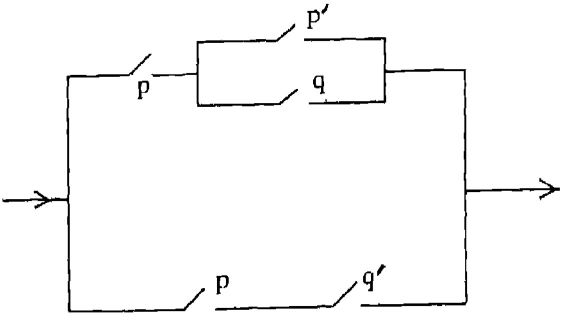


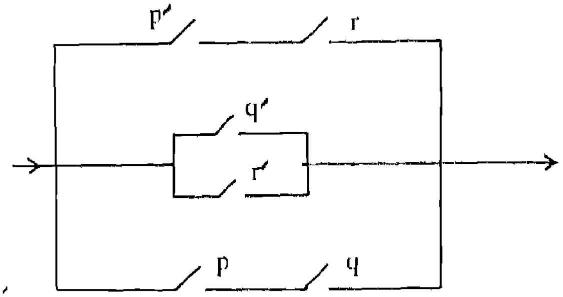
Fig. 5.12

3. Represent the each network of Fig. 5 13 symbolically and write down the truth table.



(i)

Fig. 5 13



(ii)

Fig. 5.13

5.10 Logical Equivalence

In the previous sections we dealt with statements, different in forms, but which were one and the same, i.e. have the same import or meaning. Such statements were described as equivalent. We can be more precise about the meaning of equivalent statement patterns For the purpose of logical arguments, we define two statement patterns to be *logically equivalent* if the truth tables for the two statements are identical.

Let us understand the meaning of logical implication. An implication or conditional statement, as you may recall, is a statement pattern of the form $l \rightarrow m$. Here l is called the antecedent and m the consequent. We say .

l logically implies m

if and only if the truth values of the constituents of l which make l true, make m also true. In other words the truth of l is sufficient for the truth of m.

The following examples illustrate the meaning of logical implication :

Example 5.18

Let $l : p \wedge q$ and $m : p \vee q$

If l has truth value T, necessarily p and q have truth table T (refer to the truth table of $p \wedge q$) and so $p \vee q$, i.e. m has truth value T. Thus l logically implies m.

Example 5.19

$l : p \vee \sim p$ and $m : p \rightarrow q$

As you know l is always true while m need not always be true (e.g. when p is true and q is false $p \rightarrow q$ is false) so l does not logically imply m

Example 5.20

Let $l : \sim (p \leftrightarrow q)$, $m : r$

The truth or otherwise of m does not depend on that of l. So here too l does not logically imply m.

We define two statement patterns l and m to be logically equivalent *if and only if* l logically implies m and m logically implies l.

In mathematics we have what are called necessary and sufficient conditions. For example for a triangle to be equilateral it is necessary that it is isosceles but a triangle being isosceles is not sufficient for the triangle to be equilateral.

Let us consider the following statement which characterises (fixes up) similar triangles (i.e. triangles whose corresonding sides are proportional) :

'Two triangles are similar if and only if the corresponding angles are equal'. (1)

The statement (1) is the conjunction of the two following statements or parts :

The 'if' Part : Two triangles are similar if the corresponding angles are equal. (2)

The 'only if' Part : Two triangles are similar only if the corresponding angles are equal.(3)

Let us analyse further Let

p : Two triangles are similar

q : Two triangles have corresponding angles equal.

Then you can see :

The 'if' part is $q \rightarrow p$

and the 'only if' part is $p \rightarrow q$.

Therefore the given statement is $(q \rightarrow p) \wedge (p \rightarrow q)$ which is the same as $p \leftrightarrow q$. With respect to the equivalence $p \leftrightarrow q$, $q \rightarrow p$ is called the 'sufficiency' or 'if' part; $p \rightarrow q$ is called the 'necessity' or 'only if' part.

We know that $(p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p)$. But note that what is 'sufficiency' for $p \leftrightarrow q$ is 'necessity' for $q \leftrightarrow p$ and *vice versa*

Example 5.21

Show that the statement 'If $5+2=7$, then $5+2=7$ or $7=8$'
is a logical implication.

Solution

Let p : $5+2=7$, q : $7=8$. Then the given statement is, in symbols, $p \rightarrow (p \vee q)$. Clearly, whenever the antecedent p has truth value T, the consequent has also truth value T. So the statement pattern is a logical implication.

Example 5.22

Show that the statement '$2(=2) \wedge (7=5+2)$' and '$(7=5+2) \wedge 2=2$' are logically equivalent.

Solution

This is an instance of $p \wedge q$ and $q \wedge p$ which are clearly logically equivalent statement patterns.

Remarks

(1) For two statement patterns to be logically equivalent it is *not* necessary that both contain

the same components. For example, consider the statement :

'You may come or you may not, but I will study'

If we write

p : You may come

$q = \sim p$: You may not come

r . I will study

then the given statement pattern is

$(p \vee \sim p) \wedge r$

and this statement pattern is *logically equivalent* to r i.e. $(p \vee \sim p) \wedge r \leftrightarrow r$.

To see this we note that if r is true, $(p \vee \sim p) \wedge r$ is true whether or not p is true This is because $p \vee \sim p$ is a tautology. Conversely, for the same reason, if $(p \vee \sim p) \wedge r$ is true, then r is true.

The statement pattern $p \vee \sim p$ which is a tautoloty can be considered an *extraneous statement* to r.

(2) Take the two statements $5+2=7$, $3+5=8$. Then the two statements are true simultaneously. Thus they have the same truth values. It is, however, not desirable to say that the two statements are equivalent. Thus it is reasonable to stipulate that if l, m have the same constitutent or components, have extraneous components stated in the remark above if any or *else instances (or specializations)* of logically equivalent statement patterns.

To illustrate a procedure for proving logical equivalence besides writing down the truth tables, let us consider

$l : (p \vee q) \wedge r$

$m : (p \wedge r) \vee (q \wedge r)$

Let us assume that l is true and analyse

'l is true' implies '$(p \vee q) \wedge r$ is true'
i.e. (p is true or q is true or both p, q are true) and r is true
i.e. (p and r are true or q and r are true or p, q, r are true)
(case (i)) (case (ii)) (case (iii))

Case (i) p and r are true. Therefore $p \wedge q$ true. Therefore $(p \wedge q) \vee (q \wedge r)$ true i.e. m is true.
Case (ii) q and r true. Therefore $q \vee r$ trdue. Therefore $(p \wedge q) \vee (q \wedge r)$ true i.e. m is true.
Case (iii) p, q, r are true Therefore $p \wedge q$ as well as $q \wedge r$ are true, i.e. m is true.

Thus l is true $\rightarrow m$ is true or $l \rightarrow m$.

Now let us assume that m is true and analyse

'm is true' implies '$(p \wedge r) \vee (q \wedge r)$ is true'.

i.e. '$p \wedge r$ is true or $q \wedge r$ is true or both $p \wedge r$, $q \wedge r$ are true'

i.e. 'p and r are true or q and r are true or p, q and r are true'
(case (iv)) (case (v)) (case (vi))

Case (iv) p and r true. $(p \vee q)$ is true, and r is true; $(p \vee q) \wedge r$ is true, l is true.

Case (v) q and r are true $(p \vee q)$ is true and r is true; $(p \vee q) \wedge r$ is true, i.e. l is true

Case (vi) p, q and r are true. Evidently $(p \vee q) \wedge r$ is true, i.e. l is true

Thus 'm is true' implies 'l is true' or $m \rightarrow l$.

From the above discussion, we have $l \leftrightarrow m$.

Logical equivalence is useful in proving certain assertions. Take, for instance, $x+y \neq x+z$ implies $y \neq z$. The proof of the logically equivalent statement '$x+y=x+z$' if $y=z$ is easier than the former statement. The logical equivalence follows from the fact that if

$l : p \rightarrow q$
$m : \sim q \rightarrow \sim p$,

then l and m are logically equivalent. We usually denote the logical equivalence of two statement patterns l and m by $l \equiv m$, noting, incidentally, that is an equivalence relation.

Example 5.23

Show that the following

l : If I play with you, you give me the toy
m: If you don't give me the toy then I don't play with you
n : Give me that toy, otherwise I don't play with you

are logically equivalent statements.

Solution

Let

p : I play with you.
q : You give me the toy.
Then $\sim p$: I don't play with you.
$\sim q$. You don't give me the toy.

Then l is $p \rightarrow q$, m is $\sim q \rightarrow \sim p$ while n is another verbal presentation of m. So l, m, n are logically equivalent statements.

Example 5.24

If l : $(p \rightarrow r) \vee (q \rightarrow r)$ and m : $(p \vee q) \rightarrow r$, then show that $l \neq m$.

Solution

If p and r are false and q is true then $p \rightarrow r$ is true, $q \rightarrow r$ is false and so $(p \rightarrow r) \vee (q \rightarrow r)$ is true, i.e. l has truth value T. While again $p \vee q$ is true and r is false. So $(p \vee q) \rightarrow r$ is false So m has truth value F. So l does not logically imply m. Hence $l \neq m$.

EXERCISE 5.8

1 If $l \wedge m = l \vee m$, show that $l = m$.

2. If $l \wedge m = m$, $m \wedge n = n$, then prove that $l \wedge n \equiv n$.

5.11 More about Switching Circuits and Concept of Boolean Algebra

In the earlier section we mentioned two elementary switching circuits namely connection of two switches p and q in parallel denoted by $p+q$ and in series denoted by $p \cdot q$. Let p and q be two switches in a circuit such that when p is 'on' q is 'off' and when p is 'off' q is 'on'. We call q the complement of p and denote it by p'. We also agree not to distinguish between two switches which are simultaneously on or simultaneously off and ascribe the same letter to them, i.e. p, p in Fig 5.14.

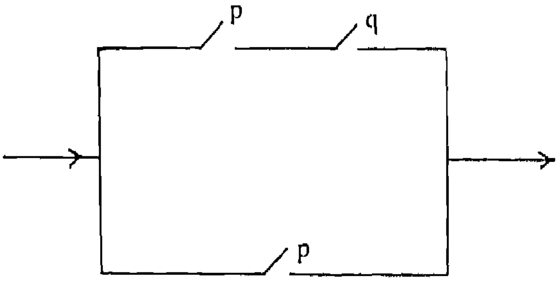
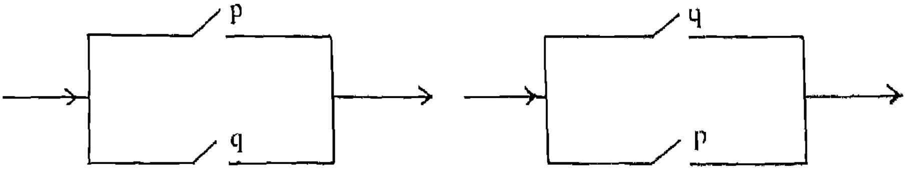


Fig. 5.14

We also write 1 for a circuit in which current flows from one terminal to the other and 0 for a circuit in which the current does not flow from one terminal to the other terminal

Look at the circuit diagrams given in Fig. 5.15 (I–X). You can easily see the truth of statements mentioned below them.

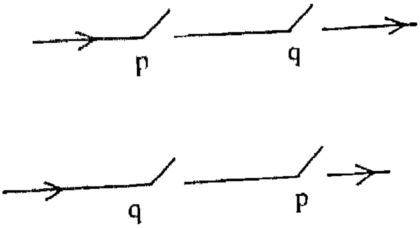


$$p + q = q + p$$

(Commutative property of '+')

(I)

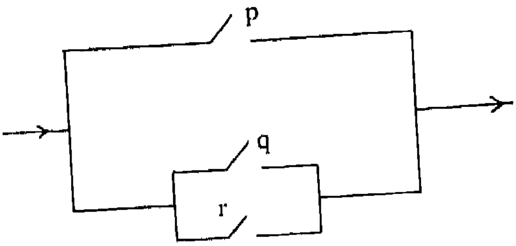
Fig. 5.15

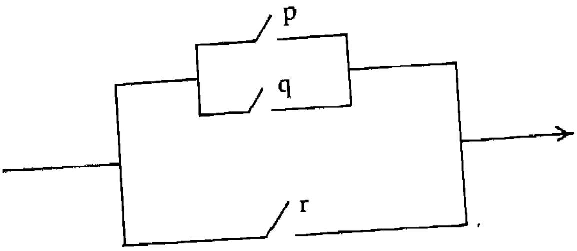


$p \cdot q = q \cdot p$

(Commutative property of '·')

(II)



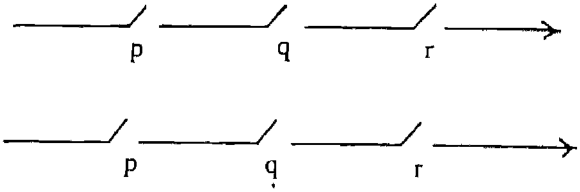


$p+(q+r)=(p+q)+r$

(Associative property of '+')

(III)

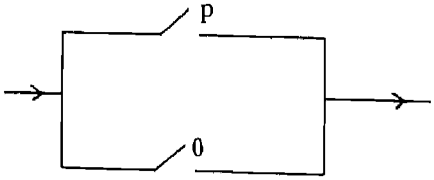
Fig. 5 15



$p \cdot (q \cdot r) = (p \cdot q) \cdot r$

(Associative property of '•')

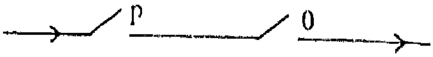
(IV)



$p+0=p$

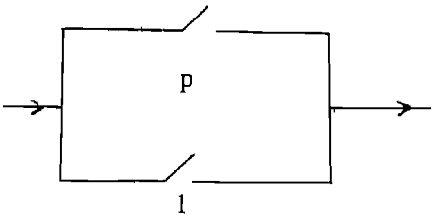
(0 is the identity element w.r to '+')

(V)

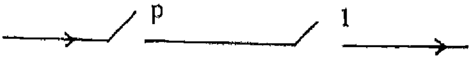


$p \; 0=0$

(VI)



$p+1=1$



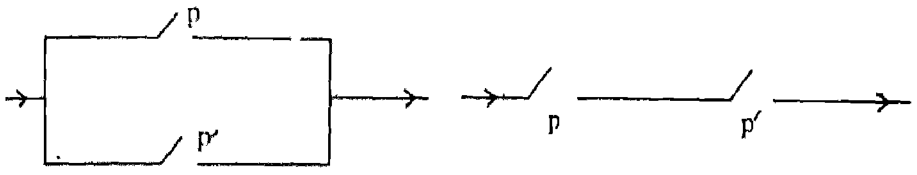
$p \cdot 1 = p$

(1 is the identity element w.r. to '•')

(VII)

(VIII)

Fig. 5.15

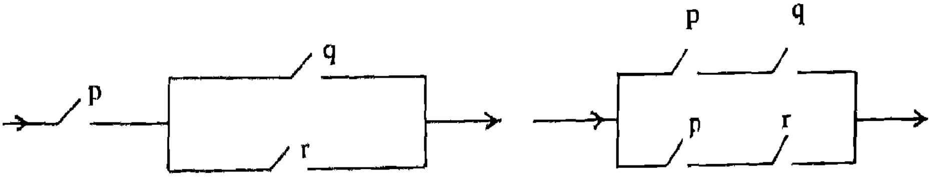


$p + p' = 1$ $p \cdot p' = 0$

(IX) (X)

Fig. 5.15

We have seen above the commutative property, associative property, identity element property w.r.t. the operations + and •. Let us see whether the distributive property also holds. Consider the following circuits shown in Fig. 5.16 (i – ii) :



$p \cdot (q + r)$ $p \cdot q + p \cdot r$

(i) (ii)

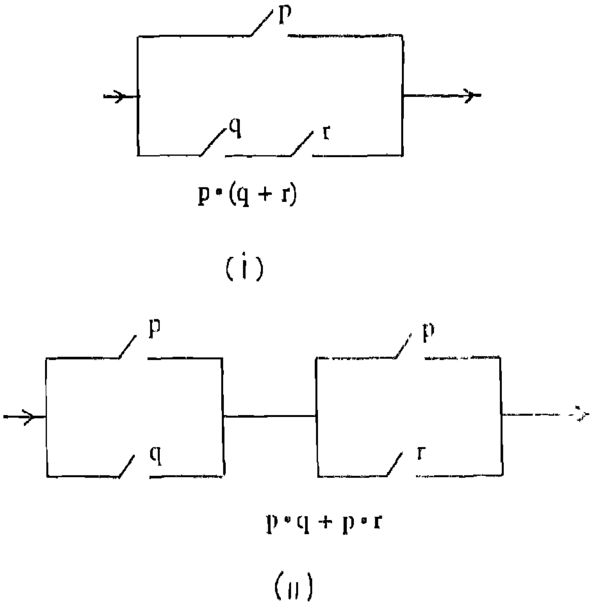
Fig. 5.16

There is flow of current in the circuit $p \cdot (q + r)$ if and only if p is on and either q or r is on. Precisely under this situation does the current flow in the circuit $p \cdot q + p \cdot r$. In other words, the above two circuits are equivalent. So we have

$$p \cdot (q + r) = p \cdot q + p \cdot r$$

This means the operation • distributes over the operation +. You may note here that the operation • distributes over the operation + in the set of integers.

Consider again the following circuits of Fig. 5.17.



$p \cdot (q + r)$

(i)

$p \cdot q + p \cdot r$

(ii)

Fig. 5.17

Note that current flows in either of the circuits if and only if either p is on or both q and r are on.

Thus we have

$p + q \cdot r = (p + q) \cdot (p + r)$

This means that the operation + also distributes over the operation•Note that the operation + *does not* distribute over the operation • in the set of integers.

Let us now summarise the properties of the set S of all switching circuits in the context of the two binary operations + and •

(i) S is closed with respect to +, i.e. if $s, t \in S$ then $s + t \in S$ (closure property)

(ii) S is closed with respect to•, i.e. if $s, t \in S$ then $s \cdot t \in S$. (closure property)

(iii) + is commutative, i e. if $s, t \in S$, then $s + t = t + s$ (commutative property)

(iv) • is commutative, i.e. if $s, t \in S$, then $s \cdot t = t \cdot s$. (commutative property)

(v) + is distributive over•, i e. if $p, q, r \in S$, then $p \cdot (q + r) = p \cdot q + p \cdot r$. (distributive property)

(vi) • is distributive over +, i.e if $p, q, r \in S$, then $p + q \cdot r = (p + q) \cdot (p + r)$ (distributive property)

(vii) + has an identity element 0, i.e. if $p \in S$, $0 \in S$ is such that $p + 0 = 0 + p = p$ (property of identity element)

(viii) • has an identity element 1, i.e. if $p \in S$, $1 \in S$ is such that $p \cdot 1 = 1 \cdot p = p$ (property of identity element)

(ix) For each element $p \in S$ there exists an element $p' \in S$ called the complement of p such that $p + p' = 1$ (identity element of the operation •) and $p \cdot p' = 0$ (identity element of the other operation +) (property of the complement)

Such a set S is called a Boolean algebra *w.r.t.* the operations +, •. Explicitly, the set S of all switching circuits is a Boolean algebra.

Next, let L be the set of all logical statements. We know that $\vee$, $\wedge$ are binary operations on L. Also $\sim$ is a unitary operation. We denote $\sim p$ by p'.

Let l be a compound statement whose components are $p, q, \ldots, z$. Obtain the statement m by changing $p, q, \ldots, z$ to $p_1, q_1, \ldots, z_1$ where p is equivalent to p_1, q to $q_1, \ldots, z$ to z_1. What can we say of m in relation to l ? Certainly l and m are equivalent in the sense that whenever l is true m is true and *vice versa*. This also means that whenever l is false m is false and *vice versa*. Hence l and m have the same truth values. Thus in any compound statement we can replace the component statements by equivalent statements without changing the truth value of the compound statement. In this sense the elements of the set L of all logical statements are to be treated as equivalence classes. With this understanding a statement which is always true and denoted by + will be the identity element for the operation $\wedge$ and a statement which is always false will be the identity for the operation $\vee$.

All the properties (i) to (ix) listed in the proceding paragraph hold good in this context too with complementation being, in this case, same as logical negation. In other words, L, the set of all logical statements, is again a Boolean algebra.

Finally, let U be a universal set and P, the set of all subsets of U, i.e. the power set of U. Let U, $\cap$ be the binary operations on P. Let complementation on P be set-theoretic complementation i.e. if $A \in T$, then $A^c = U—A$. Further U is the identity element for the operation $\cap$ and ϕ (the empty set) is the identity element for the operation U on P. The properties (i) to (ix) listed for the set S above are verified easily in this case too. Thus P, the set of all subsets of U also is a Boolean algebra.

It, therefore, follows that from any result established for any one of the three sets namely, S,L,P we can write analogous results for the other sets without the necessity of proving them again.

For example, we have the formula

$$\sim(p \vee q) = \sim p \wedge \sim q$$

in the set L of logical statements.

From this we can write the following corresponding formulae

$(A \cup B)^c = A^c \cap B^c$ in the set P.

and $(p + q)' = p' \cdot q'$ in the set S.

Moreover, the Boolean algebraic treatment of the set of all switching circuits (sometimes called networks) enables us to simplify a circuit to an equivalent circuit as the following examples show.

Example 5.25

Simplify the following network of Fig. 5.18

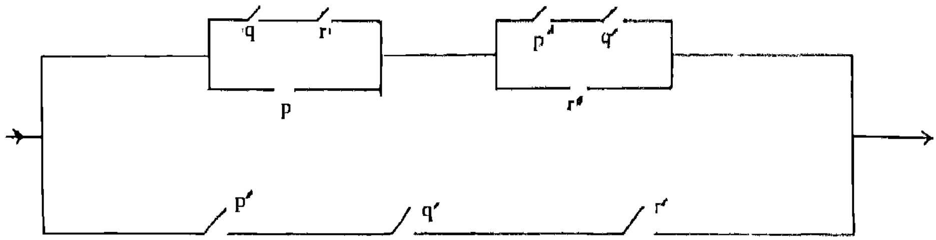


Fig. 5.18

Solution

The given circuit is

$= (q \cdot r + p) \cdot (p' \cdot q' + r') + p' \cdot q' \cdot r'$

$= q \cdot r \cdot p' \cdot q' + q \cdot r \cdot r' + p \cdot p' \cdot q' + p' \cdot q' \cdot r' + p \cdot r$ (distributive property)

$= p \cdot r' + p' \cdot q' \cdot r'$ (since $p \cdot p' = q \cdot q' = r \cdot r' = 0$)

$= (p + p' \cdot q') \cdot r'$ (distributive property)

$= [(p' + p) \cdot (p + q')] \cdot r'$ (distributive property)

$= (p + p') \cdot (p + q') \cdot r'$

$= (p + q') \cdot r'$, since $p + p' = 1$.

The simplified circuit is shown in Fig. 5.19.

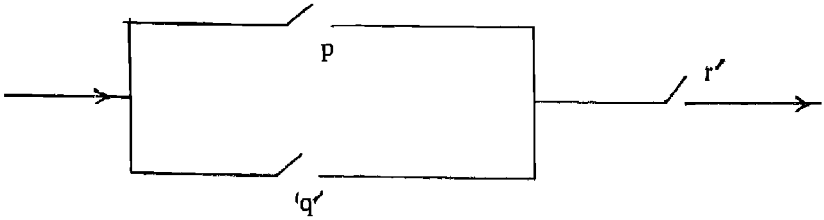


Fig. 5.19

Example 5.26

Simplify the switching circuit or network shown in Fig. 5.20.

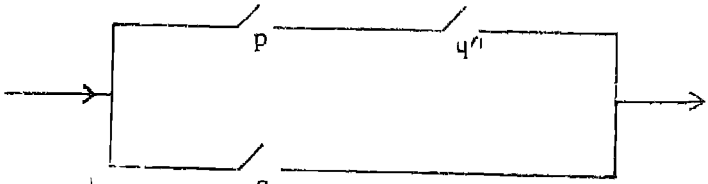


Fig. 5.20

Solution

The given network is

$p \cdot q' + q = q + (p \cdot q')$ (by commutativity of +)

$= (q + p) \cdot (q + q')$ (by distributivity of + over•)

$= q + p$ (since $q + q' = 1$).

The equivalent simpler network is therefore shown in Fig 5.21.

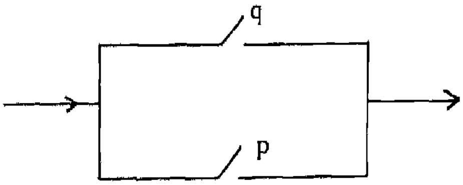


Fig 5.21

Example 5 27

Simplify the switching circuit of Fig. 5.22.

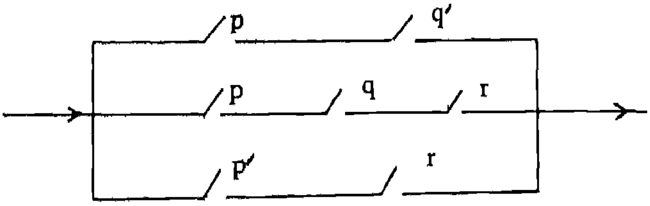


Fig. 5.22

Solution

The given circuit is

$(p \cdot q') + (p \cdot q \cdot r) + p' \cdot r$

$= p \cdot (q' + q \cdot r) + p' \cdot r$ (by distributivity of • over +)

$= p \cdot [(q + q') \cdot (q' + r)] + p' \cdot r$ (by distributivity of + over•)

$= p \cdot (q' + r) + p' \cdot r$ (since $q + q' = 1$)

$= p \cdot q' + p \cdot r + p' \cdot r$ (by distributivity of • over +)

$= p \cdot q' + (p + p') \cdot r$ (by distributivity of • over +)

$= p \cdot q' + r$ (since $p + p' = 1$)

Thus, the simplified equivalent circuit is shown in Fig. 5.23.

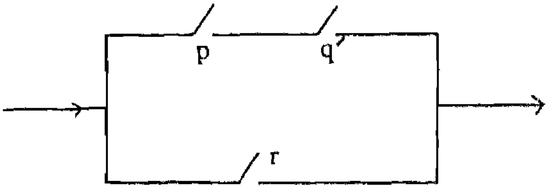


Fig. 5.23

Example 5.28

Simplify $(p+q)\cdot(q+r)$ and write down the analogous set theoretic result

Solution

$(p+q)'\cdot(q+r)' = (p'\cdot q')\cdot(q'\cdot r')$ (by DeMorgan law)

$= p'\cdot q'\cdot r'$ (by associativity of $\cdot$ and since $q'\cdot q' = q'$)

The corresponding set-theoretic result is

$$(A \cup B)' \cap (B \cup C)' = A' \cap B' \cap C'.$$

EXERCISE 5.9

1. Simplify $p+\{[(p'\cdot(p+q)]+(q\cdot r)]$.
2. Simplify the switching circuit given in Fig. 5.24.

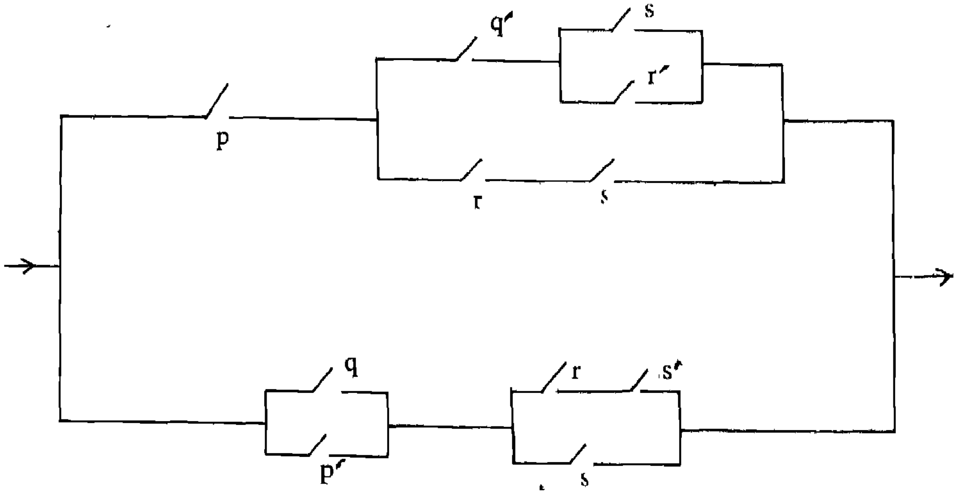


Fig. 5.24

3. Show that each element of a Boolean algebra can have only one complement.
4. Obtain the complement of the circuit given in Fig. 5.25.

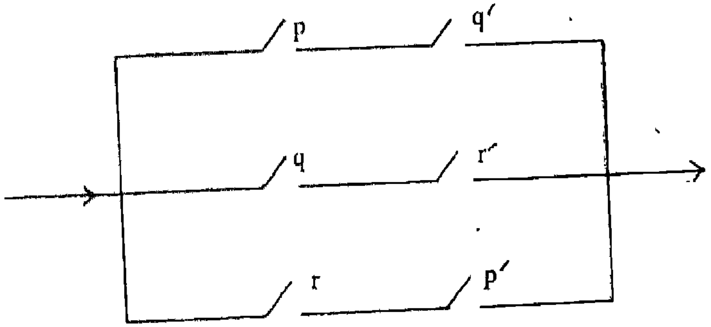


Fig. 5.25

5.12 Validity of Arguments

When coming to a conclusion the procedure adopted in mathematics or elsewhere is to have certain statements known as hypotheses which are known to be true and to deduce the truth of the conclusion when these hypotheses are simultaneously true. This really amounts to checking whether the conjunction of the hypotheses implies the conclusion. Really it is a checking of the truth of the implication under the assumption of truth for the hypotheses. If the implication is true the argument is said to be valid. If the implication is not true the argument is said to be invalid. Let us start with an example :

$$\text{hypotheses} \begin{cases} p \vee q \\ \sim p \end{cases}$$

$$\text{conclusion So} \quad q$$

Here the hypotheses $p \vee q, \sim p$ are assumed to true. If $\sim p$ is true, then p is false. Now $p \vee q$ is true. p being false q should be true if $p \vee q$ were true Thus the argument is valid. In fact $(p \vee q) \wedge \sim p \rightarrow q$ is a tautology.

Let us take another example

$$\begin{array}{l} p \rightarrow q \\ \underline{\;q\qquad} \\ \therefore p \end{array}$$

Here $p \rightarrow q$, q are assumed to be true. This does not ensure the truth of p always. For p could be false also as is seen from the truth table for $p \rightarrow q$. Thus the argument is invalid in this case.

EXERCISE 5.10

Test the validity of each argument in the following :

1. $p \leftrightarrow q$
$\sim p \vee r$
$\sim r$
So $\sim q$
2. If Ram likes mangoes, then he dislikes oranges
Ram likes oranges
So Ram dislikes mangoes.
3. If Mr. Pal is a teacher, then he is happy
If he is happy, then he does not see T.V ,
He does see T.V.
So Mr. Pal is not a teacher.
4. Assume each statement is true and draw the best possible conclusion

(i) $p \to q$
$p \vee -r$
r
$\therefore$

(ii) $p \to -q$
$\sim r \to q$
p
$\therefore$

(iii) $p \to q$
$\sim r \to -q$
$\sim r$
$\therefore$

MISCELLANEOUS EXERCISES OF CHAPTER 5

1. If p NAND $q = \sim (p \wedge q)$, p NOR $q = \sim (p \vee q)$ obtain the truth tables of NAND and NOR.
2. Obtain the truth table for the operator $\oplus$ 'exclusive' defined by $p \oplus q = (p \vee q) \wedge \sim (p \wedge q.)$
3. Prove that the following are tautologies :

(a) $(p \vee q) \wedge ((\sim r \vee \sim s) \wedge (q \to s) \vee (p \to r)) \to ((r \to p) \wedge (s \to q))$

(b) $\sim(p \leftrightarrow q) \leftrightarrow ((p \wedge \sim q) \vee (\sim p \vee q))$

4. Prove the following logical implication :

$p \wedge q \to r, p \vee q \to u, r \to \sim u$ together imply $p \to \sim q$.

5. If $p \to q$ is T, find the truth value of q, if p is (a) T, (b) F.

6. If $p \to q$ is T, $\sim q$ is T, find the truth value of p.

7. What is the truth value in each case ?
   (a) $p \wedge q$ *if* $q = t$ and t is false.
   (b) $p \wedge (q \wedge r)$ if $r = t$ and t is not true.
   (c) $p \vee (q \vee r)$ if $q = t$ and t is not false.
   (d) $p \vee q$ *if* $q \vee p$ is false.
   (e) $p \wedge (q \vee r)$ if $(p \wedge q) \vee (p \wedge r)$ is false.
8. Tell if each of the following statements is true or false, if p, q and r are any statements :
   (a) $p \vee \sim p$ is a true statement.
   (b) $p \wedge \sim p$ is false statement.
   (c) $\sim (p \wedge q) = \sim p \wedge \sim q$.
   (d) If $p \rightarrow q$ is true, then p is true.
   (e) $p \rightarrow q = q \rightarrow p$.
9. Check up whether the following arguments are logically valid (i.e. tautology) :
   (a) If today is Thursday, then yesterday was Wednesday. Yesterday was Wednesday, therefore today is Thursday.
   (b) If I do not work I will sleep. If I am worried, I will not sleep. Therefore if I am worried, I will work.
10. Prove that

$$[(p \wedge \sim q) \vee (q \wedge \sim p)] \wedge (p \vee r) = (p \vee q) \wedge (\sim q \vee \sim p) \wedge (p \vee r).$$

11. Show that

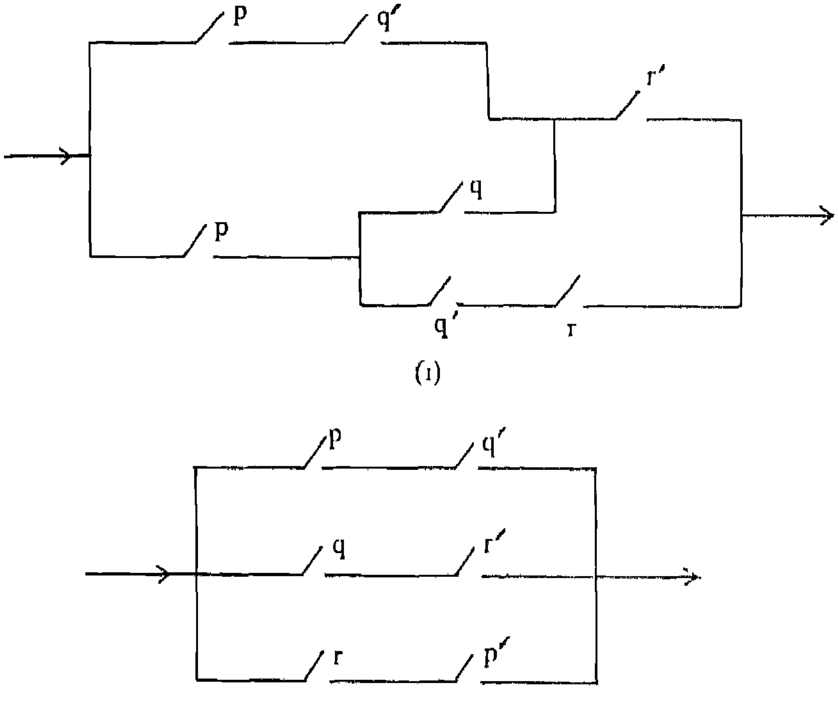
$$\sim(\sim(\sim p \wedge q) \vee \sim r) = (\sim p) \wedge q \wedge r.$$

From the following laws true for switching circuits, write the corresponding laws for logical statments and sets:

$$p \cdot (q + r) = p \cdot q + p \cdot r$$
$$p + (q \cdot r) = (p + q) \cdot (p + r)$$

12. Prove the following distributive laws :
    (i) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
    (ii) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$.
13. If p, q, r..., are elements of Boolean algebra, simplify
    (i) $[\,p + (q' \cdot r)\,] \cdot [\,p + (q' \cdot r)\,]$
    (ii) $p \cdot (p \cdot q + r)'$

14. Obtain switching circuits complementary to the circuits given in Fig. 5 26.



(i)

(ii)

Fig. 5 26

15. Factorize as for as possible :

(i) $p + q \cdot r$

(ii) $p \cdot q + r \cdot s$

if p, q, r, s are elements of a Boolean algebra.

16 Construct a switching circuit for each symbolic representation :

(a) $(p + q) \cdot p'$

(b) $p + (q \cdot r)$

(c) $(p + q) \cdot (p + r)$

(d) $(p + q') + (p + q)$

(c) $(p \cdot q) + r$

(f) $(p \cdot q) + (p' \cdot q) + (q' \cdot r)$

ANSWERS

EXERCISE 1.1

1. $2\times 6,\ 6\times 2,\ 4\times 3,\ 3\times 4,\ 12\times 1,\ 1\times 12;\ 7\times 1,\ 1\times 7$

2. (i) $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 2 & 1 & \frac{2}{3} & \frac{1}{2} \\ 3 & \frac{3}{2} & 1 & \frac{3}{4} \end{bmatrix}$

3. $x = 1,\ y = 2,\ z = 3,\ w = 4$

5. (a) Rs 15000, Rs 15000
   (b) Rs 5000, Rs 25000
   (c) Rs 25000, Rs 5000

6. Rs 157.80, Rs 167.40, Rs 281.40

7. Rs 1597.20

11. (i) $\begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$ (iii) $\begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix}$ (v) $\begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$

(vi) $\begin{bmatrix} 7 & 19 \\ 17 & 10 \end{bmatrix}$

12. (i) $\begin{bmatrix} 2a & 0 \\ 0 & 2a \end{bmatrix}$ (ii) $\begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$

(iii) $\begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix}$

13. (i) $\begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$ (ii) $\begin{bmatrix} ad-bc & bd-ab \\ 0 & d^2-bc \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 6 & 8 & 10 \\ 6 & 9 & 12 & 15 \end{bmatrix}$ (iv) $\begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$

(v) $\begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ -2 & 2 & 0 \end{bmatrix}$

(vii) $\begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$

14. (i) $\begin{bmatrix} 6 & 16 & 26 \\ -8 & -18 & -28 \end{bmatrix}$ (ii) $[\ 82\ ]$

(iii) $[a^2+b^2+c^2+d^2+ac+bd]$

(iv) $\begin{bmatrix} 0 & -1 & 1 \\ 2 & 0 & -2 \\ 5 & -2 & -3 \end{bmatrix}$

22. $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$

EXERCISE 1.2

1. (i) -57 (ii) 1 (iii) x^3-x^2+2 (iv) $\frac{1}{60}$

2. (i) $M_{11}=-1$, $M_{21}=20$, $C_{11}=-1$, $C_{21}=-20$, $|A|=-5$,
where M_{ij} = minor of a_{ij}, C_{ij} = cofactor of a_{ij},
$A=[\ a_{ij}\ ]$ and $|A|$ = determinant A

(ii) $M_{11}=-12$, $M_{21}=-16$, $M_{31}=-4$,
$C_{11}=-12$, $C_{21}=16$, $C_{31}=-4$, $|A|=40$

(iii) $M_{11}=a(b^2-c^2)$, $M_{21}=b(a^2-c^2)$, $M_{31}=c(a^2-b^2)$, $C_{11}=a(b^2-c^2)$,
$C_{21}=-b(a^2-c^2)$, $C_{31}=c(a^2-b^2)$,
$|A|=a^2(c-b)+b^2(a-c)+c^2(b-a)$

(iv) $M_{11}=5$, $M_{21}=-40$, $M_{31}=-30$, $C_{11}=5$, $C_{21}=40$, $C_{31}=-30$, $|A|=-50$

3. (i) $abc + 2fgh - af^2 - bg^2 - ch^2$ (ii) 0 (iii) $4\,abc$ (iv) 0 (v) $\lambda^2(\lambda + 3x)$ (vi) 0

EXERCISE 1.3

1. (a) 9 sq. units (b) 37.5 sq. units
(c) 23.5 sq. units (d) 15 sq. units

2. (i) $x = \frac{9}{2}, y = -\frac{7}{2}$ (ii) $x = 2, y = -\frac{2}{a}$
(iii) $x = 1, y = 3, z = -2$
(iv) $x = 1, y = 2, z = 1$
(v) $x = 3, y = 0, z = 2$
(vi) $x = 3k, y = k, z = 3k$, where k is any real (number)
(vii) $x = -2, y = -3, z = -4$
(viii) $x = 1, y = -\frac{2}{7}, z = \frac{2}{7}, w = -\frac{1}{7}$

EXERCISE 1.4

1. (i) $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ -6 & -2 & 5 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & 3 & 0 \\ -11 & 1 & 8 \\ -1 & -1 & 4 \end{bmatrix}$

2. (i) $\frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ (ii) $\frac{1}{17}\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$

(iii) $\frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$ (iv) $-\frac{1}{3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$

(v) $-\frac{1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$ (vi) $-\begin{bmatrix} -2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$

3. (i) Consistent; $x = -\frac{4}{5}, y = \frac{11}{5}$

(ii) Consistent, $x = 5 - 2k$, $y = k$, for any real number k

(iii) Inconsistent

(iv) Consistent; $x = \frac{26}{5}, y = \frac{9}{5}$

(v) Inconsistent;

(vi) Consistent; $x = \frac{5}{3}, y = k - \frac{4}{3}, z = k$, for any real number k

(vii) Consistent; $x = 1, y = 0, z = -5$

(viii) Consistent; $x = 2, y = -4, z = 7$

4. (i) $x = \frac{23}{19}, y = -\frac{32}{19}$ (ii) $x = \frac{27}{25}, y = -\frac{11}{25}$

(iii) $x = -8, y = 5$ (iv) $x = \frac{11}{24}, y = \frac{1}{24}$

(v) $x = 1, y = 2, z = -1$ (vi) $x = \frac{1}{2}, y = \frac{3}{2}, z = -1$

(vii) $x = 1, y = 2, z = 1$ (viii) $x = 2, y = 1, z = 3$

5. (i) $x = y = z = 0$ (ii) $x = y = z = 0$

(iii) $x = y = z = k$, for any real number k

(iv) $x = y = z = 0$

(v) $x = k, y = 2k, z = 3k$, for any real number k

(vi) $x = k, y = 2k, z = -k$, for any real number k

EXERCISE 2.1

1. $\mathbf{R} - \{1, 2\}$

2. $\mathbf{R} - \{n\pi : n \text{ is an integer}\}$

3. The function which associates to each real number x the unique θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan\theta = x$

4. All the numbers x such that $2 \le x < 3$. In other words, elements of the interval $[2,3)$

5. All non-negative real numbers

6. $\{1\}$

7. $\left[-\frac{1}{2}, \frac{1}{2}\right]$

8. $\left[0, \frac{3}{2}\right]$

9. $[-1,0)$ and $(0,1]$

10. $(-\infty, \infty)$

EXERCISE 2.2

3. x and $f(x)$ are always of opposite signs

4. The graph is symmetric about the line $x = 1$ parallel to the y-axis

5. The graph of g is fully above the graph of f

EXERCISE 2.3

1. The identity function itself is the function

2. sin $2x$; 2 sin x; not the same

3. e; 0; 1; $2e$

7. 0

8. 1

EXERCISE 2.4

1. 5 2. 6 3. 0 4. 2 5. -2 6. 6 7. $\frac{b}{d}$ 8. 3
9. $\frac{5}{4}$ 10. 8

EXERCISE 2.5

2. 1 **3.** $\frac{1}{2}$ **4.** 3 **5.** ± 1 **6.** $\frac{8}{3}$ **7.** 5 **8.** $\frac{5}{3}(a+2)^{\frac{2}{3}}$

EXERCISE 2.6

1. $\frac{16}{25}$ **2.** $\frac{m^2}{n^2}$ **3.** $\frac{a}{b}$ **4.** 1 **5.** 1 **6.** 1 **7.** 0 **8.** $\frac{1}{2}$ **9.** 1

EXERCISE 2.7

1. 0 **2.** 0 **6.** 0 **8.** 12

EXERCISE 2.8

1. 2 **2.** 1, No

EXERCISE 2.10

3. Non-integral points **5.** Continuous **6.** Continuous **7.** Not continuous at $x = 0$ and continuous at all other points **8.** Continuous **9.** Continuous

10. Continuous **11.** Not Continuous $\lim\limits_{x \to 0} f(x) = 0 \neq f(0)$

12. Not Continuous $\lim\limits_{x \to 0+} f(x) = 1 \neq \lim\limits_{x \to 0-} f(x) = -1$

14. (*i*) 2 (*ii*) 1

15. (*i*) $k = \frac{3}{4}$ (*ii*) $a = -2$

(*iii*) no value of m will make $h(x)$ continuous at $x = 0$

16. $a = 3$, $b = -8$

EXERCISE 3.1

1. 2 2. $2(x - 1)$ 3. 10 7. a 8. 3 9. $\frac{-2}{x^2}$ 10. $\frac{-1}{(3x + 5)^2}$

EXERCISE 3.2

1. $-3 \sin 3x$ 2. $-2 \operatorname{cosec}^2 (2x + 1)$ 3. $3ax^2 + 2bx + c$ 4. $1 - \frac{1}{x^2}$

5. $2x - 3$ 6. $e^x - 2 \sin x$

7. $\frac{2}{3}x^{\frac{1}{3}} + 4x$ 8. $2x + 3 - \frac{3}{x^2}$ 9. $\frac{1}{3}e^x$ 10. $2x - \frac{2}{x^3}$

11. $\sec^2 x + 2 \cos x - \frac{1}{2x} - e^x$ 12. $\frac{2}{x}$

EXERCISE 3.3

1. $x \cos x + \sin x$
2. $\sin x + \sin x \log x + x \log x \cos x$
3. $e^x \left(x^{-\frac{1}{3}} - \frac{1}{3}x^{-\frac{4}{3}}\right)$
4. $\sec^3 x + \sec x \tan^2 x$
5. $2x \cos x - (1 + x^2) \sin x$
6. $e^x (\cot x - \operatorname{cosec}^2 x)$
7. $\operatorname{cosec} x \,(1 - x \cot x)$

8. $5x^4 - 16x^3 + 15x^2 - 4x + 8$

9. $(1 - 2\tan x)\, 4\cos x + (5 + 4\sin x)(-2\sec^2 x)$

10. $\sin 2x$

12. $4x^3 + 3(a + c)x^2 + 2(b + d + ac)x + bc + ad$

13. $10\sec^2 x - 4\sin x + 8\cos x$

16. $e^x[x\sin x + \sin x + x\cos x]$

EXERCISE 3.4

1. $$\frac{x(\sin x + \cos x) + (1 + \cos x - \sin x)}{(x + \cos x)^2}$$

2. $$\frac{1 + \tan x \mp x\sec^2 x}{(1 + \tan x)^2}$$

3. $$\frac{1 - \log x}{x^2}$$

4. $$\frac{e^x(x - 1)}{x^2}$$

5. $$\frac{(px^2 + qx + r)(2ax + b) - (ax^2 + bx + c)(2px + q)}{(px^2 + qx + r)^2}$$

6. $$\frac{-(2ax + b)}{(ax^2 + bx + c)^2}$$

7. $$\frac{-2}{(\sin x - \cos x)^2}$$

8. $\dfrac{2\sec x\tan x}{(\sec x+1)^2}$

EXERCISE 3.5

1. $-2x\sin x^2$
2. $-\sin x\, e^{\cos x}$
3. $2\cos 2x + 8x - 20$
4. $2\cos 2x\cos 3x - 3\sin 2x\sin 3x$
5. $\dfrac{\sin 3x - 3x\cos 3x}{\sin^2 3x}$
6. $\cot x$
7. $\dfrac{\tan x}{(\log\cos x)^2}$
8. $2x\,e^{x^2}$
10. $2\cos 2x = 2(\cos^2 x - \sin^2 x)$

EXERCISE 3.6

1. $\dfrac{-1}{1+x^2}$
2. $\dfrac{1}{x\sqrt{x^2-1}}$
4. $\dfrac{1}{x\sqrt{x^2-1}}$

5. $\dfrac{-1}{x\sqrt{x^2-1}}$

6. -1

7. $\dfrac{\cos(\tan^{-1}x)}{1+x^2}$

8. $\dfrac{2}{\sqrt{1-4x^2}}$

9. $\dfrac{e^{\sin^{-1}(x+1)}}{\sqrt{1-(x+1)^2}}$

10. $\dfrac{-2\cot^{-1}x}{1+x^2}$

11. $\dfrac{-1}{\sqrt{1-x^2}}$

EXERCISE 3.7

1. $-\dfrac{x}{y}$

2. $-\dfrac{c^2}{x^2}$

3. $-\dfrac{b^2x}{a^2y}$

4. $\dfrac{2a}{y}$

5. $-\dfrac{ax + hy + g}{hx + by + f}$

6. $\dfrac{y - 4x\,(x^2 + y^2)}{4y\,(x^2 + y^2) - x}$

7. $\dfrac{ay - x^2}{y^2 - ax}$

8. $\dfrac{2xy - y^2}{2xy - x^2}$

EXERCISE 3.8

1. $(\sin x)^x\,[x \cot x + \log \sin x]$

2. $x^2 e^x \sin x \left[\dfrac{2}{x} + 1 + \cot x\right]$

3. $2^x \log 2$

4. $e^x - \cos^3 x \sin^2 x\,[1 - 3 \tan x + 2 \cot x]$

5. $(x+1)^2 (x+2)^3 (x+3)^4 \left[\dfrac{2}{x+1} + \dfrac{3}{x+2} + \dfrac{4}{x+3}\right]$

6. $2(x+1)\, e^{(x+1)^2}$

7. $x^{\sin x + \cos x}\left[\dfrac{\sin x + \cos x}{x} + \log x\,(\cos x - \sin x)\right]$

8. $(2x+3)^{x-5}\left[\dfrac{2(x-5)}{2x+3} + \log\,(2x+3)\right]$

9. $\dfrac{8^x}{x^8}\left[\log 8 - \dfrac{8}{x}\right]$

10. $\frac{1}{2}\sqrt{(x-1)(x-2)(x-3)(x-4)}\left[\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}+\frac{1}{x-4}\right]$

EXERCISE 3.9

1. $\frac{1}{t}$ **2.** $-\frac{1}{t^2}$ **3.** $\frac{b}{a\sin\theta}$

4. $-\frac{\sin t}{\cos 2t}$ **5.** $t\cos t$ **6.** $\frac{\sin\theta}{1-\cos\theta}$

7. -1 **8.** $-\left(\frac{\cos\theta+2\cos 2\theta}{\sin\theta+2\sin 2\theta}\right)$

EXERCISE 3.10

1. $\frac{2}{1+x^2}$ 2. $\frac{3}{\sqrt{1-x^2}}$ 3. $\frac{2}{1+x^2}$

4. $\frac{2}{1+x^2}$ 5. $\frac{2}{\sqrt{1-x^2}}$ 6. $\frac{1}{2(1+x^2)}$

7. $\frac{2}{\sqrt{1-x^2}}$ 8. $-\frac{1}{2\sqrt{1-x^2}}$

EXERCISE 3.11

1. 2. **2.** $6ax+2b$ **3.** $n(n-1)x^{n-2}$

4. $2\cos x - x\sin x$ **5.** $k^2 e^{kx}$

6. $-\dfrac{1 + \log x}{(x + \log x)^2}$ 7. $-\dfrac{1}{x^{\frac{3}{2}}}$

EXERCISE 3.12

1. $\cos\left(x + \dfrac{n\pi}{2}\right)$ 2. $2^n \sin\left(2x + \dfrac{n\pi}{2}\right)$

3. $a^n \cos\left(ax + \dfrac{n\pi}{2}\right)$ 4. $a^n e^{ax}$

5. $(n+1)!\,x$ 6. $n!\,a_n$ 7. $n!\,a^n$

8. 0 if $m < n$
$n!$ if $m=n$
$\dfrac{m!}{(m-n)!}\, x^{m-n}$ if $n < m$

MISCELLANEOUS EXERCISE ON CHAPTER 3

1. $x \cos x + \sin x$ 2. $x^x \left[(1 + \log x)^2 + \dfrac{1}{x} \right]$

3. $-\dfrac{c^2}{x^2}$ 4. $x^{x^x} [x^{x-1} + x^x \log x\, (1 + \log x)]$

5. $\dfrac{2xy}{3y^2 - x^2}$ 7. $-\dfrac{1}{3}$ 8. 0 9. $\dfrac{1}{x \log a}$

10. $\dfrac{x}{1 + x^2} + \tan^{-1} x$ 11. $3 \sec(3x + 4) \tan(3x + 4)$

12. $1 + (1 + \log x) \log \log x$

13. $-\left(\dfrac{1}{x}\right)^x [1 + \log x]$

14. $\dfrac{2\cos 2x - \sin 2x}{e^x}$

15. $\dfrac{2}{1+x^2}$ **16.** $-\dfrac{1}{x}\sin(\log x)$

17. $\dfrac{2}{1-x^2}$ **18.** $\sin^{m-1}x\cos^{n-1}x\,[m\cos^2x - n\sin^2x]$

19. $\dfrac{1}{\sqrt{x^2+a^2}}$ **20.** $\dfrac{5x^2-18x+12}{2\sqrt{x-3}}$

21. $\dfrac{\sec^2x}{2\sqrt{1+\tan x}}$ **22.** $\dfrac{e^x}{x}(1+x\log x)\cos(e^x\log x)$

EXERCISE 4.1

1. (i) $\frac{1}{9}\cos(.25)$ m/s (ii) 4.50 m/s (iii) 1 m/s

(iv) $\frac{4}{9}$ m/s

2. (i) $-\frac{1}{9}\sin\frac{1}{3}$ m/s^2 (ii) 2 m/s^2 (iii) 0 m/s^2

(iv) $-\frac{1}{9}$ m/s^2

3. $a = 3$, $b = -2$, $c = 6$

4. 8 km/h **5.** 110 m

EXERCISE 4.2

1. 12250 **2.** $\dfrac{u^2}{19.6}$ m **3.** 29.4 m/s

4. 39.2 m/s ; 29.4 m/s **5.** 2 s **6.** 90.2 m/s , 10.204 s ; – 9.4 m/s^2 ; 510.204 m

7. $3\frac{1}{3}$ s ; – 6 m/s^2 **8.** 0 m

EXERCISE 4.3

1. .4 m/s 2. $\frac{200\ \pi r^3}{(r+5)^2}$ km²/h

3. $\frac{7}{22}$ cm³/s 4. 900 cm³/s

5. (4,11) and $\left(-4, -\frac{31}{3}\right)$ 6. $-\frac{8}{3}$ m/s

7. $\frac{3}{8\pi}$cm/s 8. $\frac{1}{8\pi}$cm/s

EXERCISE 4.4

4. (a) Decreasing for $x < -1$ and increasing for $x > -1$

   (b) Decreasing for $x > -\frac{3}{2}$ and increasing for $x < -\frac{3}{2}$

   (c) Decreasing for $x < -2$, $x > -1$ and increasing for $-2 < x < -1$

   (d) Decreasing for $x > -\frac{9}{2}$ and increasing for $x < -\frac{9}{2}$

   (e) Decreasing for $x < 1$ and increasing for $x > 1$

8. (a) Decreasing (b) Decreasing

9. (b), (c), (d)

10. – 4

EXERCISE 4.5

1. (i) 3 (ii) 10 (iii) –2 (iv) None (v) 0 (vi) 3 (vii) 4 (viii) 2 (ix) 1

2. (i) None (ii) Min. at $x = 0$, Value = 0

   (iii) Min. at $x = 1$, Value = – 2
   Max. at $x = -1$, Value = 2

(iv) None in the interval $0<x<\pi$

(v) Max. at $x = \frac{\pi}{4}$, Value = 1

Min. at $x= \frac{3\pi}{4}$, Value = -1

(vi) Max at $x= \frac{\pi}{4}$, Value = $\sqrt{2}$

(vii) Max. at $x = \frac{3\pi}{4}$, Value = $\sqrt{2}$

Min at $x= \frac{7\pi}{4}$, Value = $-\sqrt{2}$

(viii) Max. at $x = 1$, Value = 19
Min at $x = 3$, Value = 15

(ix) Max. at $x = -2$, Value = 0
Min at $x = 0$, Value = -4

(x) Min at $x = 2$, Value = 2

(xi) Max at $x = 0$, Value = $\frac{1}{2}$

(xii) Min. at $x = 0$, Value = $(-4)^{\frac{1}{3}}$

(xiii) Max. at $x = \frac{2}{3}$, Value = $\frac{2\sqrt{3}}{9}$

(xiv) Max. at $x = 0, \frac{\pi}{2}$, Value = 1

Min. at $x = \frac{\pi}{4}$, Value = $\frac{1}{2}$

(xv) Min. at $x = -\frac{\pi}{6}$, Value = $-\frac{\sqrt{3}}{2} + \frac{\pi}{6}$

Max. at $x = \frac{\pi}{6}$, Value = $\frac{\sqrt{3}}{2} - \frac{\pi}{2}$

(xvi) Max at $x = 3$, Value = 0

(xvii) Max. at $x = 0, 1$, Value = 0

Min. at $x = \frac{3}{5}$, Value = $-\frac{54}{625}$

(xviii) Max. at $x = 0, \frac{1}{2}$ Value = 0

Min. at $x = \frac{1}{4}$, Value = $-\frac{1}{8}$

(xix) Max. at $x = -1, 1$, Value $= 0$

Min. at $x = -\frac{1}{5}$, Value $= -\frac{3456}{3125}$

5 (i) Min. $= -8$, Max. $= 8$
(ii) Min. $= 3$, Max. $= 19$
(iii) Min. $= -1.77$, Max. $= 19.625$
(iv) Min. $= -1$, Max $= \sqrt{2}$
(v) Min. $= -10$, Max. $= 8$

6. 49

7. Min. at $x = 2$, Value $= -39$
Max. at $x = 3$, Value $= 16$

8. Min. $= -55.8$, Max. $= 321$

9. $\left(\frac{\pi}{4}, 1\right), \left(\frac{5\pi}{4}, 1\right)$

10. Max $= \sqrt{2}$

11. Max $= 89$ at $x = 3$
Max $= 139$ at $x = -2$

12. $a = 120$ 13. Max $= 2\pi$, Min $= 0$

14. 12,12 15. 45, 15 16. 25,10 17. 8,8

18. 3 cm 19. $x = 5$ 21. $x = \frac{l}{2w}$

22. The cylinder with radius $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$

23. The wire is to be cut at a distance of $\frac{28\pi}{\pi + 4}$ m from one end.

EXERCISE 1.6

1. $c = 0$ 2. $c = \frac{2 \pm \sqrt{7}}{3}$

3. $c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$ **4.** $c = 0$ **5.** $c = 0$

6. $c = \frac{\pi}{4}$ **7.** $a = 11,\ b = -6,\ c = 2 \pm \frac{1}{\sqrt{3}}$

8. (a) $(0,0)$ (b) $(0,0),\ (\pi, -2),\ (2\pi, 0)$

EXERCISE 4.7

1. $c = \frac{5}{2}$ **2.** $c = \frac{1}{3}$ **3.** $c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$

4. $c = \log_2 e$

7. $c \in (a,b)$ **8.** $c \in (a,b)$

9. $\left(\frac{7}{2}, \frac{1}{4}\right)$ **10.** $\left(\frac{\sqrt{39}}{3}, \frac{13\sqrt{39}}{9}\right)$

EXERICSE 4.8

1. Tangent : $y + 10x - 5 = 0$ Normal : $x - 10y + 50 = 0$

2. Tangent : $y = 2x + 1$ Normal : $x + 2y - 7 = 0$

3. Tangent : $y = 3x - 2$ Normal : $x + 3y - 4 = 0$

4. Tangent : $y = 12x - 16$ Normal : $x + 12y - 98 = 0$

5. Tangent : $y = 0$ Normal : $x = 0$

6. Tangent : $x + y - \sqrt{2} = 0$ Normal : $y = x$

7. Tangent : $8x + 3\sqrt{5}\,y - 36 = 0$ Normal : $9\sqrt{5}\,x - 24y + 14\sqrt{5} = 0$

8. Tangent : $2x + y - 2 = 0$ Normal : $x - 2y - 6 = 0$

9. $(0,0),\ (1,2),\ (-1,-2)$

10. $2x + 3my - am^2 (2 + 3m^2) = 0$

12. $ty = x + at^2$

EXERCISE 4.9

1. .208

2. 1.96875 3. 3.9961

4. 20.025 5. .060833

6. 6 to 5 68 7. No change

8. 4π

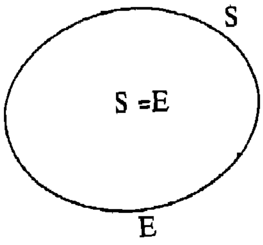
EXERCISE 5.1

1. (a) statement, (b) not a statement, (c) statement

2. (i) T (ii) F (iii) T

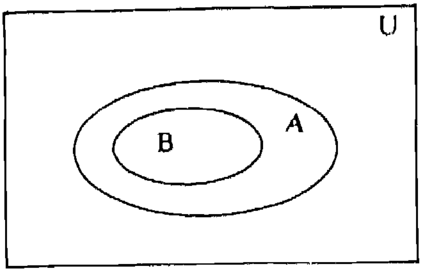
EXERCISE 5.2

1.



where S represents equilateral triangles and E represents equiangular triangles.

2.



where

U = set of all equations
A = set of all quadratic equations
B = set of quadratic equations having 2 real roots.

EXERCISE 5.3

1

p	q	$\sim p \vee q$	$\sim p \wedge \sim q$	$(\sim p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	T	F	F
T	F	F	F	F
F	T	T	F	F
F	F	T	T	T

2.

p	q	$p \rightarrow q$	$q \rightarrow p$	l	m	$l \leftrightarrow m$
T	T	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

3.

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$	$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

4.

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \to \sim p$
T	T	T	F	F
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

5.

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q \to (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

EXERCISE 5.4

1. No

2. Yes

3.

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$\sim(\sim p \wedge \sim q)$	$p \vee q$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	T	F	F

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(\sim p \vee \sim q)$	$p \wedge q$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

4. I like neither Tennis nor Foot-Ball.

EXERCISE 5.5

1.

(i)

p	q	$p \to q$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

(ii)

p	q	r	$p \to q$	$q \to r$	$(q \to r) \to (p \to q)$	$(p \to q) \to [(q \to r) \to (p \to q)]$
T	T	T	T	T	T	T
T	T	F	T	F	T	T
T	F	T	F	T	F	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

EXERCISE 5.6

1.

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$	$(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	T

2.

p	q	$p \vee q$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

3.

p	q	$\sim p$	$p \rightarrow q$	$\sim p \vee q$	$(p \rightarrow q) \leftrightarrow (\sim p \wedge q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

4.

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

EXERCISE 5.7

1.

Inputs			Outputs
p	q	r	
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	0

2.

Inputs			Outputs
p	q	p	
1	1	0	1
0	1	1	0
1	0	0	0
0	0	1	0

3.(i) $[p \bullet (p' + q)] + (p \bullet q')$

Inputs				Outputs
p	q	p'	q'	
1	1	0	0	1
1	0	0	1	1
0	1	1	0	0
0	0	1	1	0

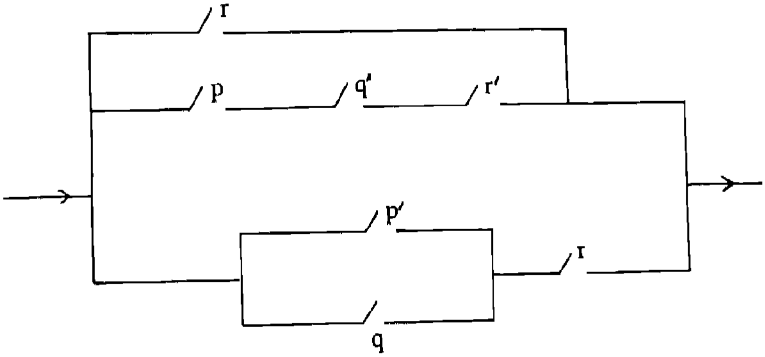
(ii) $(p' \bullet r + (q' + r') + (p \bullet q)$

Inputs						
p	q	r	p'	q'	r'	Outputs
1	1	1	0	0	0	1
1	1	0	0	0	1	1
1	0	1	0	1	0	1
1	0	0	0	1	1	1
0	1	1	1	0	0	1
0	1	0	1	0	1	1
0	0	1	1	1	0	1
0	0	0	1	1	1	1

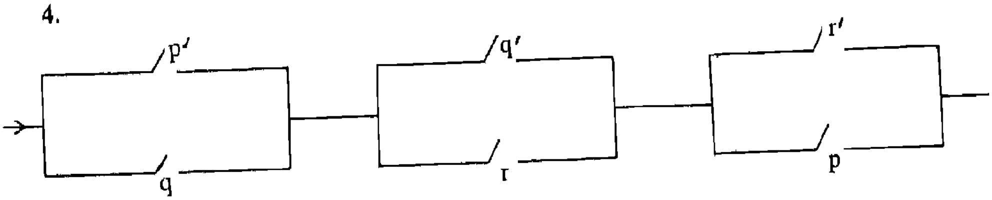
EXERCISE 5.9

1. $p + q$

2.



4.



EXERCISE 5.10

1. Valid 2. Valid

3. Valid

4. (*i*) q is true (*ii*) r is true
(*iii*) p is false

Exercise 5.11

1.

p	q	$p \wedge q$	$p \vee q$	*NAND*	*NOR*
T	T	T	T	F	F
T	F	F	T	T	F
F	T	F	T	T	F
F	F	F	F	T	T

3. (a)

p	q	r	s	$\sim r$	$\sim s$	$q \to s$	$p \to r$	$r \to p$	$s \to q$	$p \vee q$	$\sim r \vee \sim s$	$(r \to p) \wedge (s \to q)$	$(p \vee q) \wedge (\sim r \vee \sim s) \wedge (q \to s) \wedge (p \to r)$	
T	T	T	T	F	F	T	T	T	T	T	F	T	F	T
T	T	T	F	F	T	F	T	T	T	T	T	T	F	T
T	T	F	T	T	F	T	F	T	T	T	T	T	F	T
T	T	F	F	T	T	F	F	T	T	T	T	T	F	T
T	F	T	T	F	F	T	T	T	F	T	F	F	F	T
T	F	T	F	F	T	T	T	T	T	T	T	T	T	T
T	F	F	T	T	F	T	F	T	F	T	T	F	F	T
T	F	F	F	T	T	T	F	T	T	T	T	T	F	T
F	T	T	T	F	F	T	T	F	T	T	F	F	F	T
F	T	T	F	F	T	F	T	F	T	T	T	F	F	T
F	T	F	T	T	F	T	T	T	T	T	T	T	T	T
F	T	F	F	T	T	F	T	T	T	T	T	T	F	T
F	F	T	T	F	F	T	T	F	F	F	F	F	F	T
F	F	T	F	F	T	T	T	F	T	F	T	F	F	T
F	F	F	T	T	F	T	T	T	F	F	T	F	F	T
F	F	F	F	T	T	T	T	T	T	F	T	T	F	T

(b)

p	q	$p \leftrightarrow q$	$\sim (p \leftrightarrow q)$	$\sim p$	$\sim q$	$(p \wedge \sim q)$	$(\sim p \wedge q)$	$(p \wedge \sim q) \vee (\sim p \wedge q)$	$\sim (p \leftrightarrow q) \leftrightarrow (p \wedge q) \vee (\sim p \wedge q)$
T	T	T	F	F	F	F	F	F	T
T	F	F	T	F	T	T	F	T	T
F	T	F	T	T	F	F	T	T	T
F	F	T	F	T	T	F	F	F	T

2.

p	q	$p \wedge q$	$p \vee q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T	T	T	F	F
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	F	T	F

5.(a) T (b) T or F

6. F

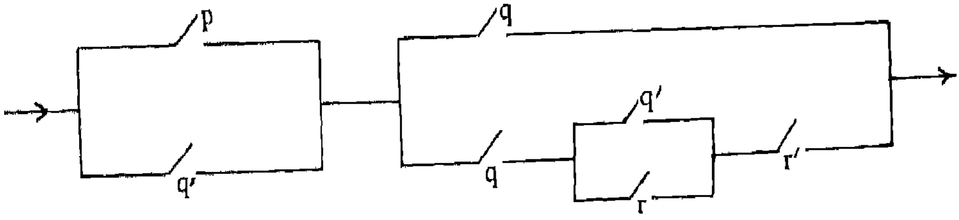
7. (a) F (b) F (c) T (d) F (e) F

8. (a) T (b) T (c) F (d) F (e) F

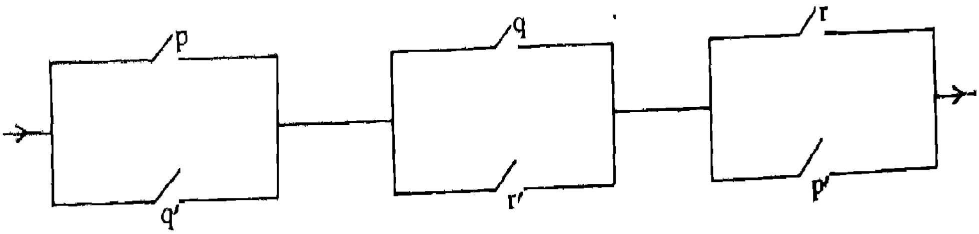
9. (a) Yes (b) Yes

13. (*i*) $p \cdot (p + r)$ (*ii*) $(p \cdot r) \cdot (p + q)$

14. (*i*)

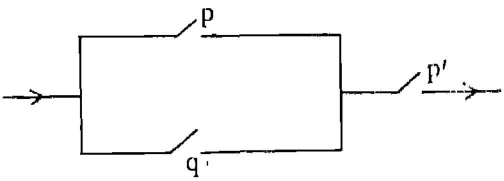


(*ii*)

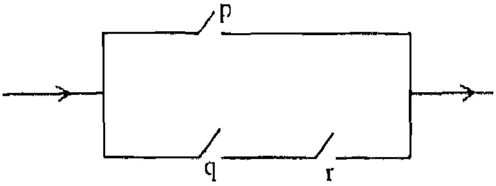


16.

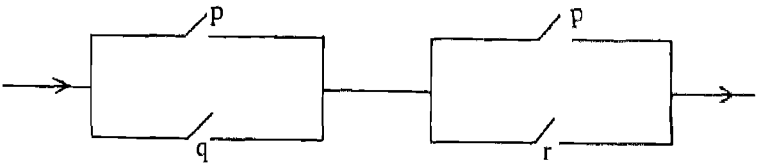
(a)



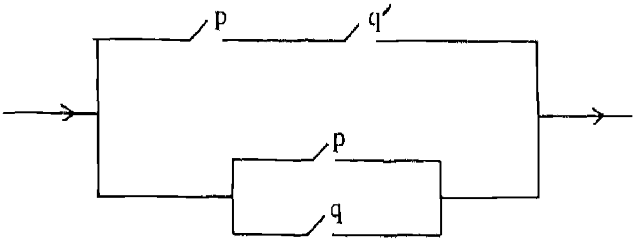
(b)



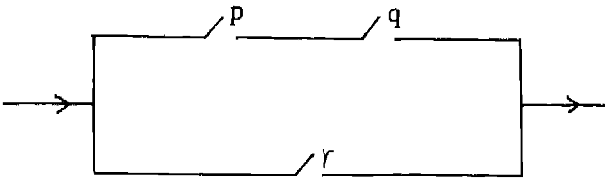
(c)



(d)



(e)



(f)